In a great many cases the methods used in the proofs of the above theorems can be used to determine whether a given continuum is a $W_n$ set. In particular, they can be used to prove that no $W_7$ set, $M$, has a complementary domain whose boundary, $J$, contains three limit points of $B(M)-J$, no $W_4$ set has a complementary domain whose boundary contains five such points, and that there exists a $W_6$ set whose outer boundary contains three such points.

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# ANOTHER REMARK ON "SOME PROBLEMS IN CONFORMAL MAPPING"

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It was remarked in [2] and proved in [3] that for every triply-connected domain $D$ there are certain triply-connected subdomains $D'$ having the same topological situation and admitting no conformal mapping into $D$ preserving this topological situation other than the identity. This result implies at once several others. Indeed let $D$ have contours $K_1$, $K_2$, $K_3$ and let the corresponding contours of $D'$ be $K_1'$, $K_2'$, $K_3'$. It is assumed no contour of $D$ reduces to a point. If $D'$ is obtained from $D$ by producing slits from $K_2$, $K_3$ out onto the same connected piece of the line of symmetry of $D$, it is clear that there is no conformal mapping of $D'$ into $D$ which can make $K_2'$ go into $K_2$ or $K_3'$ go into $K_3$ (in the natural sense of boundary correspondence). Thus for a domain $D$ and subdomain $D'$ there may exist no conformal mapping of the above type which carries either (a) a given boundary contour of $D'$ into the corresponding boundary contour of $D$ or (b) some two boundary contours of $D'$ into the corresponding two boundary contours of $D$.

The question naturally arises whether given a triply-connected domain $D$ and a triply-connected subdomain $D'$ having the same topological situation there exists a conformal mapping of $D'$ into $D$ preserving the topological situation and carrying some one contour of $D'$ into the corresponding contour of $D$. This question was raised to me by Professor A. Beurling some four or five years ago. The simple example above is not sufficient to provide an answer since in it

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$D$ and $D'$ have a common boundary. Nevertheless, the answer is in the negative as we shall now show.

First consider in $D$ the following module problem. Let $D$ lie in the $z$-plane, $z=x+iy$. Let $\Gamma$ denote the class of open arcs lying in $D$, tending to a prime end on $K_1$ in either sense, and separating $K_2$ and $K_3$ on $D$. Let $\rho$ be a non-negative real-valued function defined over $D$, of integrable square, and such that $\int_\gamma \rho|dz|$ exists, $\gamma \in \Gamma$ (possibly having the value $+\infty$), with

$$\int_\gamma \rho \, | \, dz \, | \geqq 1.$$

Let $M$ denote the greatest lower bound of $\iint_D \rho^2 dxdy$ as $\rho$ runs through the class of functions indicated above. This number is called the module of $D$ for this particular problem. It is readily seen that the greatest lower bound above is actually a minimum. Indeed if $H$ is one of the hexagons into which $D$ is divided by its line of symmetry, $H$ can be regarded as quadrangle with one pair of opposite sides given by the portion of the boundary of $H$ on $K_1$ and the portion of the line of symmetry of $D$ joining $K_2$ and $K_3$. The module of the quadrangle for the class of curves joining these sides is just twice $M$ [**2**, pp. 328, 329] and we get the extremal metric for our present problem from the extremal metric in the latter problem by a simple symmetry device [**2**, p. 348].

Let now $D'$ be a subdomain of $D$ as above and such that $K_1'$ coincides with $K_1$. Let $\Gamma'$ and $M'$ be the class of curves and module for $D'$ corresponding to $\Gamma$ and $M$ for $D$. Since every element of $\Gamma'$ is also in $\Gamma$, a standard argument [**2**, p. 329] shows that $M' \leqq M$.

In an evident fashion we can define modules for $D$ with $K_1$ being replaced by $K_2$ or $K_3$. To answer the above question in the negative it is enough to manifest a domain $D$ and subdomain $D'$ such that the three modules for $D'$ strictly exceed the three corresponding modules for $D$.

Using again the symmetry of a triply-connected domain it is enough to treat the corresponding problem for hexagons. Let $E$ be a hexagon, i.e. a simply-connected domain with six distinguished boundary elements which we call its vertices and denote by 1, 2, 3, 4, 5, 6. These mark off on the boundary the six sides of the hexagon. Let $E'$ be a second hexagon contained in $E$ with vertices $1'$, $2'$, $3'$, $4'$, $5'$, $6'$ such that $1'2'$, $3'4'$, $5'6'$ respectively lie along 12, 34, 56. The modules for $E$ corresponding to those above for $D$ are those for the three quadrangles which have the same interior as $E$ and respectively pairs of opposite sides 12, 45; 23, 56; 34, 61. In each case the module is for the class of curves joining this pair of sides. We wish to find a hexagon $E'$ contained in some such $E$ for which all three modules are larger.

For this we take $E$ to be the unit circle, its vertices in order

$$e^{i\theta},\ e^{i(2\pi/3-\theta)},\ e^{i(2\pi/3+\theta)},\ e^{i(-2\pi/3-\theta)},\ e^{i(-2\pi/3+\theta)},\ e^{-i\theta} \qquad (0 < \theta < \pi/3).$$

We shall take $E'$ to be also the unit circle, its vertices in order

$$e^{i(\theta+e)},\ e^{i(2\pi/3-\theta-e)},\ e^{i(2\pi/3+\theta+e)},\ e^{i(-2\pi/3-\theta-e)},\ e^{i(-2\pi/3+\theta+e)},\ e^{-i(\theta+e)}$$
$$(e > 0,\ \theta + e < \pi/3).$$

We now recall the well known result [1] that for a quadrangle bounded by the unit circle with verticles of affix $a$, $b$, $c$, $d$ on this circle in counterclockwise order, the module for the class of curves joining $ab$ and $cd$ is a monotone increasing function of the cross ratio of these four points taken in the form $(a-c)(b-d)/(a-d)(b-c)$.

Owing to the symmetry of the above hexagons it is enough to calculate one of the cross ratios above and we shall do it for the four points in counterclockwise order; 6, 1, 3, 4. The cross ratio is $(e^{-i\theta}-e^{i(2\pi/3+\theta)})(e^{i\theta}-e^{-i(2\pi/3+\theta)})/(e^{-i\theta}-e^{-i(2\pi/3+\theta)})(e^{i\theta}-e^{i(2\pi/3+\theta)})$. It is readily seen to have the value $\sin^2(\theta+\pi/3)/\sin^2(\pi/3)$. The corresponding value for $E'$ is $\sin^2(\theta+e+\pi/3)/\sin^2\pi/3$. This exceeds the preceding for $0<\theta<\pi/6$ for $e$ sufficiently small. Hence this provides the desired example.

This result has an interesting interpretation in terms of the domains $\tilde{N}$ and $\tilde{N}'$ associated with $E$ and $E'$ as in [2, p. 342]. Indeed we shall show that for a pair of domains $E$ and $E'$ such as those constructed above that the closure of $\tilde{N}'$ is interior to $\tilde{N}$. Using the notation of [2] we consider the mapping of the point $(a_1, a_2, a_3)$ of $\tilde{N}$ (trilinear coordinates) into the point $(\beta_1-\beta_1', \beta_2-\beta_2', \beta_3-\beta_3')$ in Euclidean 3-space.

It is clear that the image of $\tilde{N}$ cannot pass through the origin for this would give rise to a mapping of $E'$ into $E$ such that $1'2'$, $3'4'$, or $5'6'$ would coincide with 12, 34, or 56 respectively, i.e. of the type excluded. Thus we can project the image surface continuously on a sphere centred at the origin. As in [2] let $PQR$ be the triangle cut out on the surface of the sphere by the first octant.

It is proved in [2] that certain interior points of $\tilde{N}$ fall interior to $PQR$; thus if here an interior point of $\tilde{N}$ fell exterior to $PQR$ some interior point of $\tilde{N}$ would fall on the boundary of $PQR$ and thus give rise to a mapping of $E'$ into $E$ of the type excluded. Thus all interior points of $\tilde{N}$ fall interior to $PQR$.

On the other hand, no boundary point of $\tilde{N}$ falls interior to $PQR$. By the preceding none falls exterior to $PQR$, thus the boundary points of $\tilde{N}$ fall on the boundary of $PQR$. It is readily seen that the points $C, A, B$ fall on $P, Q, R$ and the sides $CA, AB, BC$ on $PQ, QR, RP$.

It follows from the above that a point interior to $\tilde{N}'$ must lie interior to $\tilde{N}$ and moreover if $\tilde{N}'$ had a boundary point in common with $\tilde{N}$ it would be mapped on a boundary point of $PQR$. Using throughout primes for entities corresponding to $E'$ we see that only $C', A', B'$ could coincide with $C, A, B$ and only a point of $C'A', A'B', B'C'$ could coincide with a point of $CA, AB, BC$. Since the latter fall on points of $PQ, QR, RP$, any of these possibilities would give rise to a mapping of $E'$ into $E$ of the type excluded. Thus the closure of $\tilde{N}'$ is interior to $\tilde{N}$ as stated.

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