$$(c[g'(z)]^* + zg'(z)) - \lambda zg'(z) = mg(z) - \lambda zg'(z).$$

Then the lemma gives $p \leqq p_1$. This completes the proof of the theorem.

REFERENCES

1. G. Ancochea, *Sur les polynomes dont les zéros son symétriques par rapport à une circonférence*, C. R. Acad. Sci. Paris vol. 221 (1945) pp. 13–15.

2. F. F. Bonsall and M. Marden, *Zeros of self-inversive polynomials*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 471–475.

3. A. Cohn, *Über die Anzahl der Wurzeln einer algebraischen Gleichung in einer Kreise*, Math. Zeit. vol. 14 (1922) pp. 110–148.

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MEASURE EXTENSIONS AND THE MARTINGALE CONVERGENCE THEOREM[1]

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1. **Introduction.** In 1940 J. L. Doob proved the following martingale convergence theorem [3].[2]

Let $\{x_n, \mathcal{F}_n, n \geqq 1\}$ be a martingale[3] with

$$\sup \{E[|x_n|]: n \geqq 1\} < \infty.$$

Then $\{x_n\}$ converges with probability 1 to a random variable x_∞ of finite expectation.

In 1946 E. S. Andersen and B. Jessen proved some limit theorems on derivatives of set functions [1]. One of the theorems is closely related to the martingale convergence theorem and is stated below.

Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n \subset \cdots$ be a nondecreasing sequence of Borel fields of subsets of a nonempty set Ω. Let P be a probability measure defined on the smallest Borel field $\mathcal{F}_\infty$ containing all the $\mathcal{F}_n$'s. Let ϕ be a bounded, countably additive set function defined on $\mathcal{F}_\infty$. Let P_n, ϕ_n be the contractions of P, ϕ to $\mathcal{F}_n$ respectively and suppose that each ϕ_n is absolutely continuous with respect to P_n. Let x_n be the derivative of ϕ_n relative to P_n. Then $\{x_n\}$ converges, except on a set of P measure 0, to

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[1] This work was done while the author was Emmy Noether Fellow of Bryn Mawr College. It is based on a portion of the doctoral thesis submitted to the University of Michigan. The thesis was written under the supervision of Professor J. L. Doob of the University of Illinois.

[2] Numbers in brackets refer to the bibliography.

[3] For the definition and properties of a martingale see [2, Chap 7].

the derivative of the P-continuous part of ϕ relative to P.

The above theorem will be designated as the A-J theorem throughout this note.

Doob has pointed out that the x_n's and $\mathcal{F}_n$'s in the A-J Theorem form a martingale. In his discussion of the relation between his martingale convergence theorem and the A-J theorem [2, appendix, pp. 630–632] the following three conditions concerning a martingale $\{x_n, \mathcal{F}_n, n \geqq 1\}$ are studied.

1. x_n's are uniformly integrable.
2. There is a countably additive bounded set function ϕ, defined on the smallest Borel field $\mathcal{F}_\infty$ containing all the $\mathcal{F}_n$'s, of which the contraction ϕ_n to $\mathcal{F}_n$ is absolutely continuous with respect to the contraction P_n of P to $\mathcal{F}_n$ and for which x_n is the derivative of ϕ_n relative to P_n for every n.
3. Sup $\{E[|x_n|]: n \geqq 1\} < \infty$.

He showed that 1 implies 2 and 2 implies 3; and the condition 2 together with the condition that ϕ be absolutely continuous with respect to P on $\mathcal{F}_\infty$ is equivalent to 1. He then demonstrated that 3 is actually weaker than 2 by exhibiting an example of a martingale which satisfies 3 but not 2. Thus he indicated that his martingale convergence theorem is more general than the A-J theorem as far as the convergence part is concerned. In this note I shall prove that if the basic space Ω on which the random variables x_n are defined is the space of real sequences $\xi = \{\xi_n\}$ and $\mathcal{F}_n$ is the smallest Borel field containing the sets of the form $\{\{\xi_n\}: \xi_1 \leqq \alpha_1, \cdots, \xi_n \leqq \alpha_n\}$ with $\alpha_1, \cdots, \alpha_n$ being any n real numbers, then 2 and 3 are equivalent. This special case is of interest because by the representation theory [2, pp. 12–15], for any martingale there is one of this type which shares most of the relevant properties of the original martingale including the convergence property. More precisely, for any martingale $\{x_n, \mathcal{F}_n, n \geqq 1\}$ where x_n's are defined on Ω with elements ω and probability measure P, there is a mapping T on Ω into the space of sequences:

$$T(\omega) = \{x_1(\omega), x_2(\omega), \cdots, x_n(\omega), \cdots\}.$$

Let $\mathcal{F}_n'$ be the Borel field of sets in the sequence space generated by the collection of sets of the form

$$\{\xi: \xi_1 \leqq \alpha_1, \xi_2 \leqq \alpha_2, \cdots, \xi_n \leqq \alpha_n\}$$

where $\alpha_1, \alpha_2, \cdots, \alpha_n$ are n real numbers. Let $\mathcal{F}'$ be the smallest Borel field containing every $\mathcal{F}_n'$ and P' be the probability measure defined on $\mathcal{F}'$ by

$$P'(\Lambda') = P(T^{-1}(\Lambda')).$$

If x_n' is the nth coordinate variable on Ω', i.e., x_n' is defined by

$$x_n'(\xi) = \xi_n,$$

then $\{x_n', \mathcal{F}_n', n \geqq 1\}$ is a martingale under the probability measure P' and $\{x_n'\}$ converges with probability 1 if and only if $\{x_n\}$ converges with probability 1. Hence the martingale convergence theorem can be deduced from the A-J theorem by applying the theorem which is to be proved.

2. **The measure extension theorem.** We shall consider a more general case of a martingale with the index set to be any subset of the real line. Let T be a set of real numbers. Let Ω be the totality of real-valued functions $\xi = \xi(t)$ defined on T.

$\mathcal{F}_t$ is the Borel field generated by the collection of sets of the form

$$[\xi : \xi(s) \leqq \alpha]$$

with $s \leqq t$ and α an arbitrary real number. $\mathcal{F}_\infty$ is the smallest Borel field containing all $\mathcal{F}_t$'s. P is a probability measure on $\mathcal{F}_\infty$. Let $\{x_t, \mathcal{F}_t, t \in T\}$ be a martingale under this probability measure and ϕ_t be a set function defined on $\mathcal{F}_t$ by

$$\phi_t(\Lambda) = \int_\Lambda x_t dP. \tag{1}$$

Then ϕ_t is bounded, countably additive, and absolutely continuous with respect to P_t, the contraction of P to $\mathcal{F}_t$. The derivative of ϕ_t relative to P_t is then x_t. Furthermore, each ϕ_t is an extension of ϕ_s if $s \leqq t$. Let ϕ be defined on $\bigcup_{t \in T} \mathcal{F}_t$ by

$$\phi(\Lambda) = \phi_t(\Lambda) \qquad \text{if } \Lambda \in \mathcal{F}_t. \tag{2}$$

Notice that $\bigcup_{t \in T} \mathcal{F}_t$ is a field of subsets of Ω and ϕ is a finite, real-valued, finitely additive set function on it.

THEOREM. *ϕ can be extended to be a countably additive set function on $\mathcal{F}_\infty$ if and only if* $\sup\{E[|x_t|] : t \in T\} < \infty$. *The extension is then bounded.*

Before proving the above theorem we shall discuss some measure preliminaries.

For a finite real-valued finitely additive set function ϕ defined on a field $\mathcal{A}$ of subsets of a set Ω the positive part ϕ^+ and the negative part ϕ^- of ϕ are defined by the following.

$$\phi^+(A) = \sup\ [\phi(B) : B \subset A,\ B \in \mathcal{A}],$$
$$\phi^-(A) = -\inf\ [\phi(B) : B \subset A,\ B \in \mathcal{A}].$$

Then ϕ^+, ϕ^- are non-negative, finitely additive set functions on $\mathcal{A}$ and

$$\phi(A) = \phi^+(A) - \phi^-(A)$$

if either $\phi^+(A)$ or $\phi^-(A)$ is finite. Furthermore, if ϕ is countably additive, then ϕ^+ and ϕ^- are also [**4**, pp. 21–22]. The following lemma concerns a necessary and sufficient condition for the existence of a countably additive extension of ϕ to the smallest Borel field $\mathcal{J}$ containing $\mathcal{A}$.

LEMMA 1. *In order that there exists a countably additive extension of ϕ to the smallest Borel field $\mathcal{J}$ containing $\mathcal{A}$, it is necessary and sufficient that the following two conditions be satisfied.*

a. *ϕ is countably additive on $\mathcal{A}$.*

b. *ϕ is bounded on $\mathcal{A}$, i.e., there is a non-negative number K for which $|\phi(A)| \leqq K$ for all $A \in \mathcal{A}$.*

PROOF. 1. *Necessity.* a is obvious. For b, let $\bar{\phi}$ be a countably additive extension of ϕ on $\mathcal{J}$. Then $\bar{\phi}$ cannot take on both values $+\infty$ and $-\infty$. Let $\bar{\phi}^+$ and $\bar{\phi}^-$ be the positive part and the negative part of $\bar{\phi}$ respectively, then $\bar{\phi} = \bar{\phi}^+ - \bar{\phi}^-$ and there are two disjoint sets C and D in $\mathcal{J}$ with $C \cup D = \Omega$ such that

$$\bar{\phi}^+(A) = \bar{\phi}(A \cap C), \qquad \bar{\phi}^-(A) = -\bar{\phi}(A \cap D)$$

for every set $A \in \mathcal{J}$ [**5**, pp. 121–123]. If $\bar{\phi}$ does not take the value $-\infty$ then

$$\bar{\phi}^-(\Omega) = -\bar{\phi}(D) < \infty.$$

Hence for every set $A \in \mathcal{J}$

$$\bar{\phi}^-(A) \leqq \bar{\phi}^-(\Omega) = -\bar{\phi}(D) < \infty,$$

and

$$\bar{\phi}^+(A) \leqq \bar{\phi}^+(\Omega) = \bar{\phi}(C) = \bar{\phi}(\Omega) - \bar{\phi}(D) < \infty$$

as $\bar{\phi}(\Omega) = \phi(\Omega) \neq \pm\infty$. Therefore both $\bar{\phi}^+$ and $\bar{\phi}^-$ are bounded. The same conclusion would be reached if $\bar{\phi}$ does not take the value $+\infty$. The boundedness of ϕ^+ and ϕ^- follows from the inequalities:

$$\phi^+(A) \leqq \bar{\phi}^+(A), \qquad \bar{\phi}(A) \leqq \bar{\phi}^-(A).$$

Since $\phi(A) = \phi^+(A) - \phi^-(A)$ for every $A \in \mathcal{A}$, ϕ is bounded.

2. *Sufficiency*. If ϕ satisfies both a and b, then ϕ^+ and ϕ^- are finite and countably additive. Since ϕ^+ and ϕ^- are non-negative they can be extended to be finite-valued countably additive measures $\overline{\phi^+}$, $\overline{\phi^-}$ on $\mathcal{F}$. For every $A \in \mathcal{A}$

$$\overline{\phi^+}(A) - \overline{\phi^-}(A) = \phi^+(A) - \phi^-(A) = \phi(A).$$

Therefore $\overline{\phi^+} - \overline{\phi^-}$ is a countably additive extension of ϕ on $\mathcal{F}$.

Using the preceding lemma we can now prove the theorem.

The ϕ defined by (1) and (2) is a finite-valued finitely additive set function. The domain of definition of ϕ is a field of sets. Suppose ϕ can be extended to be a countably additive set function on $\mathcal{F}_\infty$; by the preceding lemma there is a number K for which $|\phi(\Lambda)| \leqq K$ for all $\Lambda \in \bigcup_{t \in T} \mathcal{F}_t$. Then

$$\begin{aligned} E[\,|x_t|\,] &= \int_{[x_t \geqq 0]} x_t dP' - \int_{[x_t < 0]} x_t dP \\ &= \phi([x_t \geqq 0]) - \phi([x_t < 0]) \leqq 2K. \end{aligned}$$

Conversely, suppose $\sup\{E[|x_t|] : t \in T\} = L < \infty$; then ϕ is bounded for

$$|\phi(\Lambda)| = \left| \int_\Lambda x_t dP \right| \leqq E[\,|x_t|\,] \leqq L$$

if $\Lambda \in \mathcal{F}_t$. To show that ϕ is countably additive we shall do the following.

For each $t \in T$ define a non-negative, countably additive measure μ_t on $\mathcal{F}_t$ by the equation

$$\mu_t(\Lambda) = \int_{\Lambda t} |x_t|\, dP.$$

If $t \leqq t_1 \leqq t_2$; $t, t_1, t_2 \in T$; $\Lambda \in \mathcal{F}_t$, then $\mu_{t_1}(\Lambda) \leqq \mu_{t_2}(\Lambda) \leqq L$ for

$$\begin{aligned} \int_\Lambda |x_{t_1}|\, dP &= \int_\Lambda |E[x_{t_2} | \mathcal{F}_{t_1}]|\, dP \leqq \int_\Lambda E[\,|x_{t_2}|\,|\mathcal{F}_{t_1}] dP \\ &= \int_\Lambda |x_{t_2}|\, dP \leqq L. \end{aligned}$$

Let b be the maximum value of the closure of T (b may be infinity). Let $t_1 \leqq t_2 \leqq t_3 \leqq \cdots$ be a sequence of elements of T with $t \leqq t_n$ for every n and $\lim_{n \to \infty} t_n = b$. A set function μ_t^* is defined on $\mathcal{F}_t$ by

$$\mu_t^*(\Lambda) = \lim_{n\to\infty} \mu_{t_n}(\Lambda).$$

It is easy to see that μ_t^* is independent of the particular sequence $\{t_n\}$ chosen and is additive and finite-valued. It is also countably additive and absolutely continuous with respect to P_t, because it is the finite limit of a nondecreasing sequence of countably additive measures [5]. Furthermore, if $t<t'$; $t, t'\in T$; $\Lambda\in\mathcal{F}_t$, then

$$\mu_t^*(\Lambda) = \mu_{t'}^*(\Lambda).$$

Hence a set function μ^* can be defined on $\bigcup_{t\in T}\mathcal{F}_t$ by

$$\mu^*(\Lambda) = \mu_t^*(\Lambda)$$

if $\Lambda\in\mathcal{F}_t$. Clearly, μ^* is a non-negative finite-valued additive set function on $\bigcup_{t\in T}\mathcal{F}_t$ and is countably additive on every $\mathcal{F}_t$. Kolmogorov has proved that such a set function is also countably additive on $\bigcup_{t\in T}\mathcal{F}_t$ [6].

For each $\Lambda\in\bigcup_{t\in T}\mathcal{F}_t$,

$$|\phi(\Lambda)| \leqq \mu^*(\Lambda).$$

Hence the countable additivity of μ^* implies the countable additivity of ϕ. For, let $\Lambda_1\supset\Lambda_2\supset\cdots\supset\Lambda_n\supset\cdots$ be any decreasing sequence of sets in $\bigcup_{t\in T}\mathcal{F}_t$ for which $\bigcap_{n=1}^{\infty}\Lambda_n=$ null set, then $\lim_{n\to\infty}\mu^*(\Lambda_n)=0$. Hence $\lim_{n\to\infty}\phi(\Lambda_n)=0$ and therefore ϕ is countably additive. Q.E.D.

Bibliography

1. E. S. Andersen and B. Jessen, *Some limit theorems on integrals in an abstract set*, Det Kgl. Videnskabernes Selskab, Matematisk-Fysiske Meddelelser vol. 22, No. 14, 1946.

2. J. L. Doob, *Stochastic processes*, New York, Wiley.

3. ———, *Regularity properties of certain families of chance variables*, Trans. Amer. Math. Soc. vol. 47 (1940) pp. 485–486.

4. H. Hahn and A. Rosenthal, *Set functions*, University of New Mexico Press, 1948.

5. P. Halmos, *Measure theory*, Von Nostrand, 1950.

6. A. N. Kolmogorov, *Foundations of the theory of probability*, New York, Chelsea, 1950.

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