MINIMAL SETS OF VISIBILITY

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Let S be a set in an n-dimensional Euclidean space, E_n. The following concept was used by Horn and Valentine [2] in their study of L sets, and it provides the basis of this investigation.

DEFINITION 1. *A set $V \subset S$ is a set of visibility in S if, given any point $p \in S$, there exists a point $q \in V$ such that the closed segment $pq \subset S$.*

NOTATION. Given a point $x \in S$, let $V(x)$ denote a continuum[1] of visibility in S which contains x. The notation $V_i(x)$ will also be used.

DEFINITION 2. *The set $V(x)$ is a minimal continuum of visibility in S relative to x if, for any other continuum of visibility $V_1(x)$, we have $V_1(x) \not\subset V(x)$.*

A corresponding definition holds if we replace the word "continuum" by the words "compact convex set."

It is our purpose to investigate sets for which $V(x)$ is *unique* for each $x \in S$. The most interesting result is contained in Theorem 2. The corresponding theory in which *maximal* convex sets are considered has been developed by Strauss and Valentine [3]. The two theories are decidedly different, and this difference is explained at the end of this article.

1. Minimal compact connected sets of visibility.

THEOREM 1. *Let S be a closed set in E_n. Suppose each point $x \in S$ is contained in a unique minimal continuum of visibility $V(x)$ in S. Then either S is convex or the product $\prod_{x \in S} V(x)$ is a nonempty continuum. (Both conclusions hold if and only if S is a single point.)*

PROOF. In this and later proofs we denote the line joining x and y by $L(x, y)$.

Suppose there exists two sets $V(x)$ and $V(y)$ such that $V(x) \cdot V(y) = 0$. By Definition 1 there exists a point $q \in V(x)$ such that $yq \subset S$. Let z be the point of $V(x) \cdot yq$ which is nearest to y. The uniqueness of $V(z)$ implies that $V(z) \subset V(x)$ and that $V(z) \subset zy + V(y)$. Since $V(x) \cdot V(y) = 0$, the uniqueness of $V(z)$ together with $V(x) \cdot yz = z$ imply that $V(z) = z$. *Hence if $V(x) \cdot V(y) = 0$, S is starlike*[2] *with respect to z.*

Hence, if $V(x) \cdot V(y) = 0$, since $V(y)$ is unique, we have $V(y) = yu$

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[1] A continuum is a compact connected set.

[2] A set S is starlike if there exists a point $x \in S$ such that $V(x) = x$.

$\subset yz$, with $u \neq z$. Since from the uniqueness of $V(u)$ we have $V(u) \subset uz$, $V(u) \subset uy$, then $V(u) = u$. If $S \subset L(y, z)$, then clearly S is convex. If $S \not\subset L(y, z)$, choose any point $w \in S - L(y, z)$. Since $V(z) = z$, $V(u) = u$, we have $V(w) \subset zw$, $V(w) \subset uw$. Hence $V(w) = w$. By the same token if $p \in L(y, z) \cdot S$, then $V(p) \subset pw$, $V(p) \subset pz$, whence $V(p) = p$. Thus for any point $a \in S$, we have $V(a) = a$. Hence, if $V(x) \cdot V(y) = 0$, the set S is convex.

Now, assume S is not convex. Hence, for any $x \in S$, $y \in S$, we must have $V(x) \cdot V(y) \neq 0$. Choose $z \in V(x) \cdot V(y)$. Since $V(z) \subset V(x)$, $V(z) \subset V(y)$, we have $V(z) \subset V(x) \cdot V(y)$. Since for any set $V(a)$ we have $V(a) \cdot V(z) \neq 0$, it follows that $V(a) \cdot V(x) \cdot V(y) \neq 0$. By a simple induction it follows that *every finite* collection of the sets $\{V(x), x \in S\}$ has a nonempty intersection. Hence, by the usual compactness argument, we have $\prod_{x \in S} V(x) \neq 0$, if S is not convex.

Finally, to prove $\prod_{x \in S} V(x)$ is connected if S is not convex, we first prove $V(x) \cdot V(y)$ is connected. Suppose this were not so, and let K_1 and K_2 be two components of $V(x) \cdot V(y)$. Since K_1 and K_2 are each connected closed sets of visibility in S, each contains a minimal closed connected set of visibility. Hence, as proved above, we must have $K_1 \cdot K_2 \neq 0$ if S is not convex. The fact that $\prod_{x \in S} V(x)$ is connected follows by a simple induction together with the fact that if every finite subcollection of a collection of continua have a connected intersection, then they all have a connected intersection. This completes the proof of Theorem 1.

2. Minimal compact convex sets of visibility.

In this section we confine ourselves to sets $S \subset E_2$.

LEMMA 1. *Let S be a compact set in E_2. Suppose each point $x \in S$ is contained in a unique minimal closed convex set of visibility $V(x)$ in S. Then S is simply connected.*[3]

PROOF. Suppose S is not simply-connected, and let K be a bounded component of the complement of S. Let $H(K)$ be the convex hull of $\bar{K}$, where $\bar{K}$ is the closure of K. Let $B(H)$ denote the boundary of $H(K)$. There exists a point $x \in B(H)$ such that a unique line of support L to $H(K)$ at x exists. If $x \in \bar{K}$, let $x = y$. If $x \notin \bar{K}$, let L_1 be the line through x perpendicular to L, and let y be the point of $L_1 \cdot \bar{K}$ which is nearest to x. Since $H(K)$ is bounded, there exists a unique line of support L^* to $H(K)$ which is parallel to L, and distinct from L. Clearly since K is an open connected set, $y \cdot L^* = 0$. Let $L^* \cdot \bar{K} \equiv G$.

[3] A set in E_2 is simply-connected if each component of its complement is unbounded.

To prove that G is a single point, suppose there exist two points $u \in G$, $v \in G$. Let a be any point between u and v on L^*, and let $L(a)$ be the line through a perpendicular to L^*. Let b be the point of $\overline{K} \cdot L(a)$ which is nearest to a. The line segment of S which joins b to a point of $V(y)$ and the segment ab (degenerate or not) violates the connectedness of K, since u and v are limit points of K. Hence, $L^* \cdot \overline{K} = L^* \cdot B(H) = p$, a point of S. Moreover, the line L^* is not a unique line of support to $H(K)$ at p, otherwise $V(y)$ would not be visible from p.

Now, let L_i be a sequence of parallel lines between L and L^* such that $L_i \rightarrow L^*$ as $i \rightarrow \infty$. Choose $r_i \in L_i \cdot B(K)$, $s_i \in L_i \cdot B(K)$ such that the segment $r_i s_i$ contains the set $L_i \cdot \overline{K}$, and such that in terms of a direction on L, L^*, and L_i we have $r_i < s_i$ on L_i. Due to the position of the point y, defined above, the visibility of $V(r_i)$ and $V(s_i)$ implies that $V(r_i)$ and $V(s_i)$ must intersect L on opposite sides of x relative to L. In fact, $V(r_i) \cdot L$ and $V(s_i) \cdot L$ have the same order on L as r_i and s_i have on L_i. Since $L^* \cdot B(H) = p \in S$, it follows that $r_i \rightarrow p$, $s_i \rightarrow p$ as $i \rightarrow \infty$. Each of the collections $\{V(r_i)\}$ and $\{V(s_i)\}$ contains a convergent subsequence which converges to a closed convex set of visibility V_r and V_s respectively, with $p \in V_r$, $p \in V_s$. Let R_+ be the closed half-plane bounded by L^* which does *not* contain the point x. Since $V_r \cdot L \neq 0$, $V_s \cdot L \neq 0$, with x between $V_r \cdot L$ and $V_s \cdot L$, and since $p \in B(K)$, it follows that $V_r \cdot V_s \subset R_+$. On account of the uniqueness of $V(p)$, we have $V(p) \subset V_r$, $V(p) \subset V_s$. Hence, $V(p) \subset V_r \cdot V_s \subset R_+$. However, due to the position of the point y, there exists no point $q \in V(p)$ such that $yq \subset S$ (K is an open connected set). This is a contradiction; hence, S is simply connected.

LEMMA 2. *Assume the same hypotheses about S as in Lemma 1. Suppose there exists two points x and y in S such that $V(x) \cdot V(y) = 0$. Then S is starlike.*

PROOF. A line L divides the plane into two closed half-planes, denoted by R_+ and R_-. A *mutually separating* line of support to $V(x)$ and $V(y)$ is one which is a line of support to each, and one for which either

$$V(x) \subset R_+, \quad V(y) \subset R_- \quad \text{or} \quad V(x) \subset R_-, \quad V(y) \subset R_+.$$

If $V(x)$ and $V(y)$ are not collinear, there exist two mutually separating lines of support to $V(x)$ and $V(y)$, denoted by L_1 and L_2. If $V(x)$ and $V(y)$ are collinear, then $L_1 = L_2$. If $L_1 \neq L_2$, let $p = L_1 \cdot L_2$. If $L_1 = L_2$, choose $p \in L_1$ between x and y, with $p \notin V(x)$, $p \notin V(y)$. Let $r_i \in L_i \cdot V(x)$, $s_i \in L_i \cdot V(y)$ $(i = 1, 2)$. Since $V(y)$ is a minimal set of visibility, there exist points $p_1 \in V(y)$, $p_2 \in V(y)$ such that $r_1p_1 \subset S$, $r_2p_2 \subset S$. The quadrilateral $r_1p_1p_2r_2$ (degenerate or nondegenerate) may be simple or not, but in any case its sides all belong to S. Since L_1 and L_2 are mutually separating lines of support to $V(x)$ and $V(y)$, it is easily seen that triangle $r_1r_2p \subset r_1p_1p_2r_2$. Since, by Lemma 1, S is simply-connected, we must have triangle $r_1r_2p \subset S$. Hence the convex hull $H[p+V(x)] \subset S$. In exactly the same manner, we have $H[p+V(y)] \subset S$. Since $V(p) \subset H[p+V(x)]$, $V(p) \subset H[p+V(y)]$, and since $H[p+V(x)] \cdot H[p+V(y)] = p$, the uniqueness of $V(p)$ implies $V(p) = p$, so that S is starlike.

The following definition is due to Brunn [1].

DEFINITION 3. *The set* $K(S) \equiv \{x \in S,\ V(x) = x\}$ *is called the Kerneigebiet of* S. (*The set* S *is starlike relative to each point of the Kerneigebiet.*)

THEOREM 2. *Let* S *be a compact set in* E_2, *and suppose each point* $x \in S$ *is contained in a unique minimal closed convex set of visibility* $V(x)$ *in* S. *Then either* S *is convex or* S *is starlike with respect to one and only one point of* S. (*In other words, the Kerneigebiet* $K(S)$ *is either* S *or it is a single point of* S.)

PROOF. Suppose S is not starlike. Then by Lemma 2, for each pair of points $x \in S$, $y \in S$ we have $V(x) \cdot V(y) \neq 0$. Then by exactly the same argument as given in Theorem 1, involving the finite intersection property and compactness, we must have $\prod_{x \in S} V(x) \neq 0$. But this is a contradiction, since $\prod_{x \in S} V(x) \subset K(S)$. Hence, S is starlike. Suppose there exist two distinct points $a \in K(S)$, $b \in K(S)$. If $S \subset L(a, b)$, then S is a line segment. If $z \in S - L(a, b)$, the uniqueness of $V(z)$ implies $V(z) \subset za$, $V(z) \subset zb$. However, this implies $V(z) = z$ so that $z \in K(S)$. Similarly, if $c \in [L(a, b) - a - b] \cdot S$, then $V(c) \subset cz$, $V(c) \subset ca$, so that $V(c) = c$. Hence, $K(S) = S$ if $a \neq b$. Thus, either $K(S) = S$ or $K(S)$ is a single point of S. This completes the proof of Theorem 2.

There exist a variety of interesting examples of the set S in Theorem 2. For instance, the set consisting of two externally tangent circular disks is a nonconvex one containing interior points.

The corresponding theory for *unbounded* closed sets $S \subset E_2$ offers considerably more difficulty. Although I am able to establish a nontrivial generalization of Theorem 2 when at least *one* of the sets $V(x)$ is bounded, the case when *all* the $V(x)$ are unbounded remains unsettled.

3. **Concluding remarks.** In a previous paper [3] Straus and Valentine proved the following theorem.

"Let S be a closed connected set in a finite dimensional linear space, and let R_n be the subspace of minimal dimension which contains S. Then the set S is convex if and only if each point $x \in S$ is contained in a unique *maximal* convex subset of S of dimension greater than or equal to $n-1$."

Observe that the notion of *visibility* is not required in the above uniqueness requirement. This cannot be done for *minimal* convex sets of visibility since a minimal convex set of S containing a point x is always x. This is the reason the theory in this paper differs essentially from that used by Straus and Valentine.

The generalization of Theorem 2 to E_n ($n > 2$) remains unsettled, and it appears to offer considerable difficulties. Finally, the converse of Theorem 2 is clearly false. For instance, a circular disk together with two outward normals (segments) is an obvious counterexample.

Bibliography

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