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TORONTO, ONTARIO

ON SEMI-SIMPLE LIE ALGEBRAS

JEAN DIEUDONNÉ

1. The primary purpose of this note is to give a new proof for the sufficiency of E. Cartan's criterion for semi-simplicity of a Lie algebra, namely that its Killing form be nondegenerate. My proof differs from the usual ones in the fact that it uses no result from the theory of nilpotent Lie algebras, and is valid for a base field of *arbitrary characteristic*.

2. Let $\mathfrak{g}$ be a Lie algebra over a field K, having finite dimension $n>0$. A symmetric bilinear form $\phi(X, Y)$ over $\mathfrak{g}\times\mathfrak{g}$ is called *invariant* if $\phi([X, Y], Z)=\phi(X, [Y, Z])$ identically. This is the case for the *Killing form* $\mathrm{Tr}(\mathrm{ad}(X)\ \mathrm{ad}(Y))$, where $\mathrm{ad}(X)$ is the endomorphism $Y\rightarrow[X, Y]$ of the vector space $\mathfrak{g}$. It is well known that when the Killing form of $\mathfrak{g}$ is nondegenerate, $\mathfrak{g}$ does not contain any abelian ideal $\neq(0)$ (one has only to remark that if $\mathfrak{a}$ is such an ideal, and $A\in\mathfrak{a}$, then $(\mathrm{ad}(A)\ \mathrm{ad}(X))^2=0$ for any $X\in\mathfrak{g}$, by an elementary computation, hence $\mathrm{Tr}(\mathrm{ad}(A)\ \mathrm{ad}(X))=0$ for all $X\in\mathfrak{g}$). E. Cartan's criterion is therefore a consequence of the more general result:[1]

THEOREM. *If the Lie algebra $\mathfrak{g}$ does not contain any abelian ideal $\neq(0)$, and if there exists a symmetric invariant nondegenerate bilinear form $\phi(X, Y)$ on $\mathfrak{g}\times\mathfrak{g}$, then $\mathfrak{g}$ is a direct sum of simple nonabelian subalgebras.*

Let $\mathfrak{m}$ be a minimal ideal in $\mathfrak{g}$; as $[\mathfrak{m}, \mathfrak{m}]$ is an ideal of $\mathfrak{g}$, contained in $\mathfrak{m}$, $[\mathfrak{m}, \mathfrak{m}]$ is either (0) or $\mathfrak{m}$; but the first case is excluded, since

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[1] I am indebted to N. Jacobson for calling my attention to this generalization, as well as for simplifying my original proof.

then $\mathfrak{m}$ would be abelian, therefore $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{m}$. Let $\mathfrak{m}'$ be the subspace of $\mathfrak{g}$ orthogonal (for ϕ) to $\mathfrak{m}$; as ϕ is invariant, $\mathfrak{m}'$ is an *ideal* in $\mathfrak{g}$, for the relations $X \in \mathfrak{m}$, $Y \in \mathfrak{m}'$, $Z \in \mathfrak{g}$ imply $\phi(X, [Z, Y]) = \phi([X, Z], Y) = 0$ since $[X, Z] \in \mathfrak{m}$. The intersection $\mathfrak{m} \cap \mathfrak{m}'$ can only be (0) or $\mathfrak{m}$, since $\mathfrak{m}$ is minimal; let us show that the second case cannot occur, in other words that the relation $\mathfrak{m} \subset \mathfrak{m}'$ cannot hold. Indeed, we would then have $\phi(X, Y) = 0$ for any elements X, Y of $\mathfrak{m}$. But if $A \in \mathfrak{m}$, one can write $A = \sum_i [B_i, C_i]$, where B_i and C_i are in $\mathfrak{m}$. Then for every $X \in \mathfrak{g}$, $\phi(A, X) = \sum_i \phi([B_i, C_i], X) = \sum_i \phi(B_i, [C_i, X]) = 0$, since $[C_i, X] \in \mathfrak{m}$; but this contradicts the assumption that ϕ is nondegenerate. Therefore $\mathfrak{m} \cap \mathfrak{m}' = (0)$; as ϕ is nondegenerate, $\mathfrak{g}$ is the direct sum of the two ideals $\mathfrak{m}$ and $\mathfrak{m}'$. But the restriction to $\mathfrak{m}' \times \mathfrak{m}'$ of the form ϕ is then a symmetric invariant nondegenerate bilinear form, and $\mathfrak{m}'$ cannot contain any abelian ideal $\neq (0)$, for such an ideal would also be an ideal in $\mathfrak{g}$. Induction on the dimension of $\mathfrak{g}$ completes therefore the proof.

UNIVERSITY OF MICHIGAN