A CONDITION THAT $\lim_{n\to\infty} n^{-1}T^n = 0$

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In this note we prove the following theorem:

Let T be a bounded linear transformation on a Banach space; let the spectrum of T be interior to the unit circle, with the possible exception $z=1$; further, suppose that there is an $M>0$ and an $\eta>0$ such that if z is in the resolvent set for T, $|z|\geqq 1$, and $|z-1|\leqq\eta$, then $\|(z-1)\cdot(z-T)^{-1}\|\leqq M$. Then $\lim_{n\to\infty} n^{-1}T^n=0$.

As a consequence of a theorem of Dunford,[1] if T obeys the above hypothesis, the sequence $f_n(T)=n^{-1}\sum_{r=1}^{n}T^r$ will then fail to converge to a projection if and only if $z=1$ is a nonisolated spectral point of T.

To prove the theorem, let $\epsilon>0$ be chosen. Let $\eta'>0$ such that the length of the arc intercepted on the unit circle by the circle of radius η', center at $z=1$, is less than $2\pi M^{-1}\epsilon$.[2] Let $\delta=\min(\eta,\eta')$ and Γ_δ the circle with radius δ. Γ_δ intersects the unit circle in two points z_0, $\bar z_0$. To be definite, let $\operatorname{Im}(z_0)>0$. Let Γ' be the arc of the unit circle which does not contain $z=1$. Let $\bar\Gamma_\delta$ be the arc of Γ_δ not interior to the unit circle. Let N be such that if $n>N$,

$$\left\| n^{-1}\int_{\Gamma'} z^n(z-T)^{-1}dz\right\| < 2\pi\epsilon, \tag{1}$$

$$\left\| n^{-1}\int_{\bar\Gamma_\delta} (z-T)^{-1}dz\right\| < 2\pi\epsilon, \tag{2}$$

$$n^{-1} < M^{-1}(e-1)^{-1}\epsilon. \tag{3}$$

Let $n>N$ and, in what follows, hold n fixed. Let $\delta'=\min(n^{-1},\delta)$, and $\Gamma_{\delta'}$ the circle of radius δ'. $\Gamma_{\delta'}$ intersects the unit circle in z' and $\bar z'$, $\operatorname{Im}(z')>0$. Let Γ_+ and Γ_- be respectively the arcs of the unit circle from z_0 to z' and $\bar z_0$ to $\bar z'$, and not exterior to Γ_δ. Let $\bar\Gamma_{\delta'}$ be the arc of $\Gamma_{\delta'}$ not interior to the unit circle. Then,

$$\int_{\bar\Gamma_\delta} n^{-1}z^n(z-T)^{-1}dz = \left(\int_{\bar\Gamma_{\delta'}}+\int_{\Gamma_+}+\int_{\Gamma_-}\right) n^{-1}z^n(z-T)^{-1}dz. \tag{4}$$

Now,

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[1] N. Dunford, *Spectral theory. I. Convergence to projections*, Trans. Amer. Math. Soc. vol. 54 (1943) pp. 185–217.

[2] In all the circles we construct, the center will be at $z=1$.

$$(5)\quad \int_{\Gamma_-} n^{-1}z^n(z-T)^{-1}dz = n^{-1}\sum_{j=0}^{n-1}\int_{\Gamma_-} z^j(z-1)(z-T)^{-1}dz + n^{-1}\int_{\Gamma_-}(z-T)^{-1}dz,$$

$$(6)\quad \int_{\Gamma_+} n^{-1}z^n(z-T)^{-1}dz = n^{-1}\sum_{j=0}^{n-1}\int_{\Gamma_+} z^j(z-1)(z-T)^{-1}dz + n^{-1}\int_{\Gamma_+}(z-T)^{-1}dz,$$

$$(7)\quad \begin{aligned}\int_{\bar\Gamma_{\delta'}} n^{-1}z^n(z-T)^{-1}dz &= n^{-1}\sum_{j=1}^{n} C_{j,n}\int_{\bar\Gamma_{\delta'}}(z-1)^{j-1}(z-1)(z-T)^{-1}dz \\ &\quad + n^{-1}\int_{\bar\Gamma_{\delta'}}(z-T)^{-1}dz.\end{aligned}$$

The sum of the last terms in the right members of (5), (6), and (7) is $n^{-1}\int_{\bar\Gamma_\delta}(z-T)^{-1}dz$, and is less, in norm, than $2\pi\epsilon$ by (2).

$$\left\| n^{-1}\sum_{j=0}^{n-1}\int_{\Gamma_+} z^j(z-1)(z-T)^{-1}dz\right\| \leqq M\text{ (length of }\Gamma_+) \leqq 2\pi\epsilon.$$

A similar statement can be made for the first term of the right member of (5).

Finally,

$$\begin{aligned}&\left\| n^{-1}\sum_{j=1}^{n} C_{j,n}\int_{\Gamma_{\delta'}}(z-1)^{j-1}(z-1)(z-T)^{-1}dz\right\| \\ &\qquad \leqq n^{-1}M2\pi\sum_{j=1}^{n} C_{j,n}(\delta')^j = n^{-1}M2\pi[(1+\delta')^n - 1] \\ &\qquad \leqq n^{-1}M2\pi[(1+n^{-1})^n - 1] < n^{-1}M2\pi(e-1) < 2\pi\epsilon \text{ by (3).}\end{aligned}$$

But $\|n^{-1}T^n\| = \|(2\pi i)^{-1}(\int_{\Gamma'}+\int_{\bar\Gamma_\delta})n^{-1}z^n(z-T)^{-1}dz\|$,[3] and using (1) and the inequalities obtained above, this is less than 5ϵ. The theorem is proved.

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[3] Dunford, loc. cit.