ON THE SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

CHOY-TAK TAAM

1. Let $P(x)$ and $Q(x)$ be complex-valued Lebesgue-measurable functions defined for all non-negative x, the functions $1/P(x)$ and $Q(x)$ being of the class $L(0, R)$ for every positive R. A solution of the differential equation

$$(P(x)W')' + Q(x)W = 0 \tag{1.1}$$

is an absolutely continuous function $W(x)$ such that $P(x)W'(x)$ is equal almost everywhere to an absolutely continuous function $W_1(x)$, say, and that

$$W_1'(x) + Q(x)W = 0 \tag{1.2}$$

is satisfied for almost all x. In the sequel only those solutions which are distinct from the trivial solution $(\equiv 0)$ shall be considered.

On the positive x-axis let I be an interval which need not be closed or bounded. The equation (1.1) will be called disconjugate on I if and only if no solution of (1.1) possesses more than one zero on I.

It is the purpose of this note to derive a general criterion (Theorem 1) for the differential equation (1.1) disconjugate on an interval and from which to prove a comparison theorem (Theorem 2). These results generalize those obtained previously by the author for the case $P(x)=1$ [2, Theorems 1 and 9]. When $P(x)=1$ and $Q(x)$ is real, an interesting discussion of disconjugate differential equations was given by A. Wintner [4].

The method of proof of Theorem 1 is a modification of that employed in [2, Theorem 1].

2. Write

$$P(x) = p_1(x) + ip_2(x), \qquad Q(x) = q_1(x) + iq_2(x), \tag{2.1}$$

where p_1, p_2, q_1 and q_2 are real. We first prove the following general criterion.

THEOREM 1. *Suppose that the following conditions are satisfied:*

(1) *$m = m(x)$ is a real-valued function absolutely continuous on every closed subinterval of I,*

(2) *for some real constants j and k, jp_1+kp_2 is positive on I and $1/(jp_1+kp_2)$ belongs to the class L on every closed subinterval of I,*

Presented to the Society, April 25, 1953; received by the editors March 16, 1953.

(3) *$m(x)$ satisfies the inequality*

$$m' + m^2/(jp_1 + kp_2) \leqq -(jq_1 + kq_2) \tag{2.2}$$

almost everywhere on I.

Then (1.1) *is disconjugate on I. Furthermore, if I is closed at least at one end, there is a solution of* (1.1) *which does not vanish on I.*

PROOF. Suppose that the theorem is not true. Then there is a solution $W(x)$ which has at least two zeros a and b, $a<b$, in I. We shall show that this leads to contradiction.

Let W_1 be the absolutely continuous function which is equal to PW' almost everywhere on I. Write

$$W = u + iv, \qquad W_1 = u_1 + iv_1, \tag{2.3}$$

where u, v, u_1, and v_1 are real. It is clear that

$$u_1 = p_1u' - p_2v', \qquad v_1 = p_2u' + p_1v'. \tag{2.4}$$

Separating the real and imaginary parts of (1.2), we get

$$u_1' = -q_1u + q_2v, \qquad v_1' = -q_2u - q_1v. \tag{2.5}$$

The equalities in (2.4) and (2.5) hold almost everywhere on I. Let

$$L = j(uu_1 + vv_1) + k(uv_1 - u_1v) - m(u^2 + v^2). \tag{2.6}$$

Differentiating (2.6) and simplifying the result with (2.4) and (2.5), we have

$$\begin{aligned} L' = {} & (jp_1 + kp_2)(u'^2 + v'^2) - 2m(uu' + vv') \\ & - (m' + jq_1 + kq_2)(u^2 + v^2) \end{aligned} \tag{2.7}$$

almost everywhere on I. Completing the squares, (2.7) yields

$$\begin{aligned} L' = {} & (jp_1 + kp_2)\left[(u' - mu/(jp_1 + kp_2))^2 \right. \\ & \left. + (v' - mv/(jp_1 + kp_2))^2\right] \\ & - \left[m' + m^2/(jp_1 + kp_2) + jq_1 + kq_2\right](u^2 + v^2). \end{aligned} \tag{2.8}$$

The first term on the right-hand side of (2.8) is positive almost everywhere on $[a, b]$, otherwise u and v would be solutions of the differential equation

$$y' = my/(jp_1 + kp_2) \tag{2.9}$$

on $[a, b]$, and, since u and v vanish at a, u and v must vanish identically on $[a, b]$, but this is impossible owing to the fact that $W \not\equiv 0$. Integrating both sides of (2.8) from a to b and using (2.2), we have clearly

$$L(b) - L(a) > 0. \tag{2.10}$$

Since L vanishes at a and b, we have contradiction. This proves that W cannot possess two zeros on I and hence (1.1) is disconjugate on I.

If I is closed at the left end with end point a, then the argument above shows that $L(x) \geqq L(a)$ for all x on I. Since $jp_1 + kp_2$ is positive, j and k cannot both be zero. Suppose that j is not zero. Let W be a solution with

$$W(a) = 1, \qquad W_1(a) = [m(a) + 1]/j.$$

For this solution it is easy to verify that $L(a) = 1$. Hence $L(x) \geqq 1$ for all x on I. Consequently, from (2.6), this solution does not vanish on I. The cases that $j = 0$, $k \neq 0$, and I is closed at the right end can be proved similarly. This completes the proof of Theorem 1.

3. In this section, we shall prove a comparison theorem. Consider another differential equation

$$(r(x)y')' + f(x)y = 0, \tag{3.1}$$

where r and f are real-valued functions defined for all non-negative x, r being positive, and $1/r$ and f belonging to $L(0, R)$ for every positive R. On the positive x-axis, let I_0 be an interval which is either closed or open, and if open need not be bounded.

THEOREM 2. *Suppose that the following conditions are satisfied*:

(1) (3.1) *is disconjugate on* I_0,

(2) *for some real constants* j *and* k, *the inequalities* $jp_1 + kp_2 \geqq r$, $jq_1 + kq_2 \leqq f$ *hold almost everywhere on* I_0.

Then (1.1) *is disconjugate on* I_0. *Furthermore, if* I_0 *is closed, there is a solution of* (1.1) *which does not vanish on* I_0.

PROOF. It is known that if (3.1) is disconjugate on I_0, there exists a real-valued function $m(x)$ which is absolutely continuous on every closed subinterval of I_0 and satisfying the inequality

$$m' + m^2/r \leqq - f \tag{3.2}$$

almost everywhere on I_0 [3, Theorem 1]. From (3.2) and condition (2) of the theorem, it is clear that

$$m' + m^2/(jp_1 + kp_2) \leqq - (jq_1 + kq_2) \tag{3.3}$$

holds amost everywhere on I_0. The theorem then follows from Theorem 1.

4. In the following theorem, we consider the differential equation

$$((jp_1 + kp_2)y')' + (jp_1 + kp_2)^{-1}G^2y = 0, \tag{4.1}$$

where

$$G(x) = 2\int_a^x (jq_1 + kq_2 + g)dx + A. \tag{4.2}$$

THEOREM 3. *Suppose that the following conditions are satisfied*:

(1) *j, k and A are real constants,*

(2) *$g = g(x)$ is real-valued, non-negative on $[a, b]$ and belongs to $L(a, b)$,*

(3) *$jp_1 + kp_2$ is positive on $[a, b]$ and $(jp_1 + kp_2)^{-1}$ belongs to $L(a, b)$,*

(4) (4.1) *is disconjugate on $[a, b]$.*

Then (1.1) *is disconjugate on $[a, b]$.*

PROOF. Since (4.1) is disconjugate on $[a, b]$, according to [3, Theorem 1], there exists a real-valued function $n(x)$ absolutely continuous on $[a, b]$ and satisfying

$$n' + n^2/(jp_1 + kp_2) \leqq -G^2/(jp_1 + kp_2) \tag{4.3}$$

almost everywhere on $[a, b]$. Let $m = (n - G)/2$. Using (4.2) and (4.3), it is easy to verify that m satisfies (2.2) almost everywhere on $[a, b]$. The theorem then follows from Theorem 1.

Theorem 3 can be easily modified to apply to an open interval, bounded or unbounded.

Theorem 3 is a generalization of a theorem due to P. Hartman [1].

REFERENCES

1. P. Hartman, *On linear second order differential equations with small coefficients*, Amer. J. Math. vol. 73 (1951) pp. 955–962.

2. C. T. Taam, *Non-oscillation and comparison theorems of linear differential equations with complex-valued coefficients*, Portugaliae Mathematica vol. 12 (1953) pp. 57–72.

3. ———, *Non-oscillatory differential equations*, Duke Math. J. vol. 19 (1952) pp. 493–497.

4. A. Wintner, *On the non-existence of conjugate points*, Amer. J. Math. vol. 73 (1951) pp. 368–380.

THE CATHOLIC UNIVERSITY OF AMERICA