ON AN OPEN QUESTION CONCERNING FIXED POINTS

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A space X is said to have the f.p.p. (fixed point property) if every continuous function f from X to X has a fixed point. Whether if X and Y have the f.p.p. then $X\times Y$ has the f.p.p. is an open question.

A space X is said to have the F.p.p. (fixed point property for multi-valued functions) if every continuous multi-valued function F from X to X has a fixed point, i.e., a point x such that $x\in F(x)$. Interest in fixed points for multi-valued functions leads one to question under what conditions on the spaces X and Y and on the multi-valued function F on X to Y there will exist a continuous trace f of F, that is, a continuous function f on X to Y such that $f(x)\in F(x)$ for all x. For some specific multi-valued functions F it is possible to produce a continuous trace. It is by use of these traces that most of the fixed point theorems in the literature for multi-valued functions are proved. In fact the open question mentioned above (which is concerned only with single-valued functions) can be answered if one can produce continuous traces of two particular multi-valued functions. This paper proves some fixed point theorems by producing continuous traces, shows that a continuous multi-valued function need not have a continuous trace, and gives an example which indicates that a general theorem on the existence of a continuous trace is not likely to be established without strong conditions on F regardless of what conditions are placed on X and Y. This example answers in the negative the generalization of the above open question to the multi-valued case, exhibits a continuous multi-valued function which has no continuous trace, shows that the general Tychonoff cube does not have the F.p.p., and shows that a space with the f.p.p. need not have the F.p.p.

NOTATION. By $\{x_a\}\to x_0$ we denote a sequence of points indexed by a directed set A and converging to x_0. The directing relation in A will be denoted by $*$.

DEFINITION 1. *Continuous.* A multi-valued function on a space X to a space Y is said to be continuous at x_0 if $\{x_a\}\to x_0$ implies that $F(x_0)=$cofinal limit $\{F(x_a)\}=$residual limit $\{F(x_a)\}$. F is said to be

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continuous if it is continuous at every x in X.

For the details of how this definition of continuity is related to the definitions used elsewhere in the literature see [1]. This definition is strong enough, however, to insure that the example is valid for functions continuous under the definitions used by Ratner [2], Wallace [3], Eilenberg and Montgomery [4], Kakutani [5], Banach and Mazur [6], and Michael [7].

Directly from the definition of trace we establish the following

LEMMA 1. *Let f be a trace of a multi-valued function F on X to Y and let x be a fixed point of f. Then x is a fixed point of F.*

From Lemma 1 it is clear that a sufficient condition for a continuous multi-valued function F to have a fixed point is that F have a continuous trace which has a fixed point. But this is not a necessary condition. In fact one obtains a continuous multi-valued function with a fixed point and no continuous trace by defining F from the unit circle at the origin in the complex plane to itself by $F(z) =$ the two square roots of z.

The example.

THEOREM 1. *A bounded closed interval I of real numbers has the* F.p.p.

PROOF. Brouwer's theorem assures us that I has the f.p.p. Hence in view of Lemma 1 it is sufficient to prove that every continuous function F on I to I has a continuous trace. We shall in fact prove that if R is a bounded closed interval of real numbers and F is a continuous multi-valued function on a space X to R, then F has a continuous trace f. Define f on X to R by $f(x) = \text{lub } \{y \mid y \in F(x)\}$.

It is known [1] that a multi-valued function from a space X to a compact Hausdorff space Y is continuous if and only if $x_0 \in X$ implies:

(1) $F(x_0)$ is closed,

(2) V open containing $F(x_0)$ implies that there exists an open set U' containing x_0 and such that whenever $x \in U'$ then $F(x) \subset V$, and

(3) $y_0 \in F(x_0)$, $y_0 \in V$, and V open imply that there is an open set U'' containing x_0 such that whenever $x \in U''$ then $F(x) \cap V \neq 0$.

Let $V_{2\phi}$ be an open interval of length 2ϕ with center $f(x')$, where ϕ is a positive real number. Then V_ϕ is also an open set containing $f(x')$. By (3) there is an open set U'' containing x_0 such that $x \in U''$ implies that $F(x) \cap V_\phi \neq 0$. Hence $x \in U''$ implies that lub $\{y \mid y \in F(x)\} = f(x) \geqq f(x') - \phi$.

Let $V = \{y \mid y < \phi + f(x')\}$. This set V is open containing $F(x')$. Hence (2) implies that there exists an open set U' containing x' such that $x \in U'$ implies that $F(x) \subset V$. Then $x \in U'$ implies $f(x) = \text{lub } \{y \mid y \in F(x)\} \leqq f(x') + \phi$. Let $U = U' \cap U''$. Then $x \in U$ implies that $|f(x) - f(x')| \leqq \phi$, therefore $f(x) \in V_{2\phi}$, and f is continuous at x'.

Theorem 1 established that I has the F.p.p. Let r be a continuous function which retracts the unit square $I \times I$ onto the unit disc X. Define F from X to X as follows. If x is the origin, let $F(x) = S$, where S denotes the unit circle with center at the origin. If x is not the origin: (a) Extend the segment from the origin through x until it meets S in a point A. (b) Draw a perpendicular at x to the radius constructed in (a) and denote its intersections with S as B and C. (c) Consider the closed arc BAC on S. Let $MBACN$ be the closed arc of S with center A, length twice the length of the arc BAC, and having end points M and N. (d) Let $F(x) = MBACN$. That F has the three properties utilized in the proof of Theorem 1 is geometrically evident and hence F is continuous. Define G to be iFr followed by a rotation of ninety degrees, where i denotes the injection of X into $I \times I$. The continuity of G follows from [1, Proposition 18]. Now $x \in (I \times I) - S$ implies that $x \notin G(x)$ because $G(I \times I) \subset S$. Also $x \in S$ implies that $F(x) = x$ and the rotation moves x. Hence G has no fixed point and consequently $I \times I$ does not have the F.p.p.

Cartesian and apex functions. Most theorems on fixed points for multi-valued functions demand either that $F(x)$ be a connected set for every x or that $F(x)$ be a convex set for every x. The literature appears to be void of fixed point theorems with no condition on the image of a point. Theorem 1 above is such a theorem. The example above shows that the two-dimensional cube does not have the F.p.p. A simple extension of this result shows that no Tychonoff cube of dimension greater than one has the F.p.p. Then in order to prove theorems concerning fixed points for functions on a Tychonoff cube one must place some further conditions on F. The following theorems indicate that F may enjoy much greater pointwise freedom than is allowed under the usual assumption that $F(x)$ is either convex or connected.

Definition 2. *Cartesian function.* Let $T = PI_a$ be a Tychonoff cube. A subset Y_0 of T is called a T-cartesian subset if $Y_0 = PM_a$, where M_a is a subset of I_a. A function F from a space X to a space Y is called a cartesian function if there exists a homeomorphism h of Y into some Tychonoff cube T such that (1) $h(Y)$ is a retract of T and (2) $x \in X$ implies that $hF(x)$ is a T-cartesian subset of T.

Theorem 2. *Every continuous Cartesian function from a space X to itself has a fixed point.*

PROOF. Let F be a continuous cartesian function on X to itself. There is a Tychonoff cube $T=PI_a$, a homeomorphism h of X into T, and a retraction r of T onto $h(T)$ such that $x\in X$ implies that $hF(x)$ is a T-cartesian subset of T. Let $G(x)=ihF(x)$, where i denotes the injection of $h(X)$ into T. Define H on T to T by $H(x)=Gh^{-1}r(x)$ $=ihFh^{-1}r(x)$ and define H_a from T to I_a by $H_a(x)=$ the projection of $H(x)$ in I_a. That H and H_a are continuous is established in [1]. In the proof of Theorem 1 it was shown that H_a has a continuous trace f_a. Kakutani [5] showed that if each f_a is a continuous single-valued function, so is f defined by $f(x)=Pf_a(x)$. For each $a, f_a(x)$ is an element of $H_a(x)$ and $H(x)=PM_a(x)$, where M_a is a subset of I_a, so that $f(x)$ $\in H(x)$. Now f is a continuous single-valued function from a Tychonoff cube to itself and hence has a fixed point.

Let x' be a fixed point of H. Then $x'\in H(x')=ihFh^{-1}r(x')$ $=hFh^{-1}r(x')$. Since $x'\in T$, $r(x')\in h(T)$ and hence there exists $x''\in T$ such that $r(x')=h(x'')$. Then $x'\in H(x')=hFh^{-1}h(x'')=hF(x'')$ $\subset hF(X)\subset h(X)$. The function r retracts T onto $h(X)$, hence $x'=r(x')$ $=h(x'')$, $x''=h^{-1}(x')$, $x'\in hF(x'')$, and $x''=h^{-1}(x')\in F(x'')$.

DEFINITION 3. *Apex set.* Let B be a closed subset of a Tychonoff cube $T=PI_a$. Denote by B_a and b_a the projections of B and of b, respectively, in I_a. For a fixed a denote lub $\{b_a | b_a\in B_a\}$ by $m(B_a)$. The set B_a is closed and hence $m(B_a)\in B_a$. If there is only one point in B which projects onto $m(B_a)$ we say that B is an apex subset of T with respect to a.

DEFINITION 4. *Apex function.* A function F from a space X to a space Y is called an apex function if there is a homeomorphism h of Y onto a retract of a Tychonoff cube $T=PI_a$ such that, for some fixed $a=a(1)$, $x\in X$ implies that $hF(x)$ is an apex subset of T with respect to $a(1)$.

THEOREM 3. *Every continuous apex function from a space X to itself has a fixed point.*

PROOF. Let G be a continuous apex function from X to X. Then there is a Tychonoff cube $T=PI_a$, a homeomorphism h of X into T, a retraction r of T onto $h(X)$, and a fixed $a=a(1)$ such that $x\in X$ implies that $hG(x)$ is an apex subset of T with respect to $a(1)$. The function $F=hGh^{-1}r$ is defined on T to T. As in the proof of Theorem 2, $F_{a(1)}$ is continuous. By (1) in the proof of Theorem 1, $t\in T$ implies that $F_{a(1)}(t)$ is closed and hence $t\in T$ implies that $m[F_{a(1)}(t)]$ is an element of $F_{a(1)}(t)$. The hypothesis that G is an apex function implies that there is exactly one $t'\in F(t)$ such that t' projects onto $m[F_{a(1)}(t)]$. Define a single-valued function f from T to T by $f(t)=t'$.

Clearly f is a trace of F.

Let x^0 be an element of T and let y^0 be an element of $F(x^0)$. Let ϕ be a real number greater than zero and let $V_{a(1)}(\phi)$ contain $f_{a(1)}(x^0) = y^0_{a(1)}$. By (3) in the proof of Theorem 1 there is an open set U'' containing x^0 such that whenever $x \in U''$, then $F_{a(1)}(x) \cap V_{a(1)}(\phi^0) \neq 0$, and hence $f_{a(1)}(x) = \text{lub } \{y_{a(1)} \mid y_{a(1)} \in F_{a(1)}(x)\}$ is greater than $y_{a(1)} - \phi$.

Let $W = \{y_{a(1)} \mid y_{a(1)} < y^0_{a(1)} + \phi\}$. Then W is open and contains $F_{a(1)}(x^0)$ so that (2) in the proof of Theorem 1 implies the existence of an open set U' containing x^0 such that $F(x)$ is contained in W for all $x \in U'$, i.e., $x \in U'$ implies lub $\{y_{a(1)} \mid y_{a(1)} \in F_{a(1)}(x)\} < y^0_{a(1)} + \phi$.

Now $x \in U' \cap U''$ implies that $y^0_{a(1)} - \phi < y_{a(1)} = f_{a(1)}(x) < y^0_{a(1)} + \phi$, and hence $f_{a(1)}$ is continuous.

Let $f(x^0) = y^0$. Assume that f is not continuous at x^0. Then there exists an open set W containing y^0, a directed set D, and a sequence $\{x_d\} \to x^0$ such that $d \in D$ implies that $f(x_d) \notin W$. The set $T - W$ is closed and hence compact. The sequence $\{f(x_d)\}$ determines a net ϕ on D to $T - W$ defined by $\phi(d) = f(x_d)$ and hence [9, Theorem 24] there exists a subnet (E, θ) with a limit $\bar{y}$ in $T - W$. Let k be the function on E to D satisfying the definition of subnet. Since $\phi(d) = f(x_d)$, $\theta(e) = \phi k(e) = f(x_{k(e)})$. If V is an open set containing $\bar{y}$, then there exists $e(V) \in E$ such that $e * e(V)$ implies that $\theta(e) \in V$. If U is an open set containing x^0, then there exists d' such that $d * d'$ implies that $x_d \in U$. If $d' \in D$ then there exists e' such that $e * e'$ implies that $k(e) * d'$. Therefore $e * e'$ implies that $x_{k(e)} \in U$. Then $\{x_{k(e)}\} \to x^0$ and $\{f_{a(1)}(x_{k(e)})\} \to \bar{y}_{a(1)}$. It was shown that $f_{a(1)}$ is continuous, hence $f_{a(1)}(x^0) = \bar{y}_{a(1)}$. But $f_{a(1)}(x^0) = y^0_{a(1)}$, therefore $\bar{y}_{a(1)} = y^0_{a(1)}$. Then the hypothesis that F is an apex function implies that $\bar{y} = y^0$, which is an element of W. But $\bar{y} \in T - W$. This contradiction implies that our assumption was false and f is continuous. This trace f is a continuous single-valued function on a Tychonoff cube to itself and hence Lemma 1 implies that F has a fixed point x_0. The proof that $h^{-1}(x_0)$ is a fixed point of G is a reiteration of the last statement in the proof of Theorem 2.

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ERRATA, VOLUME 3

C. W. Curtis, *A note on noncommutative polynomials.*

p. 965, line 10 from the bottom. Add to condition (b): "where $T(r)\neq 0$ if $r\neq 0$."

ERRATA, VOLUME 4

W. R. Mann, *Mean value methods in iteration.*

p. 507, Display (2) should include the following:

$$\lim_{i\to\infty} a_{ij} = 0 \qquad \text{for all } j.$$

E. Michael, *A note on paracompact spaces.*

p. 835, diagram near the top of the page. For "covering" read "open covering" (twice), and for "refinement" read "open refinement" (twice).