A NOTE ON PREHARMONIC FUNCTIONS

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1. Let L be the set of points whose coordinates are rational integers. Let D be a domain, that is to say, an open connected set, and let G be the set $D \cdot L$. A point $P(m, n)$ of G is an *interior* point if the four points $(m \pm 1, n)$, $(m, n \pm 1)$ contiguous to P belong to G. Otherwise P is a *boundary* point.

A function $f(m, n)$ defined on G is preharmonic if the value of f at any interior point is the mean of the values of f at the contiguous points, that is to say

$$4f(m, n) = f(m + 1, n) + f(m - 1, n) + f(m, n + 1) + f(m, n - 1).$$

For several decades the subject of preharmonic functions has been considered by many mathematicians, and the connection with harmonic functions has long been known. A recent paper by Heilbronn [1] states a number of theorems which are the analogues of classical theorems for harmonic functions.

In this note we consider functions which are preharmonic and nonnegative in the half-plane $n \geqq 0$ and prove a representation theorem analogous to that for positive harmonic functions [2], and a theorem which is the analogue of the Phragmén-Lindelöf type theorem for positive harmonic functions [3; 4].

2. We require the following lemmas:

LEMMA 1. *If $f(m, n)$ is preharmonic on a bounded domain D, then $f(m, n)$ is either constant or attains its maximum and minimum on D on the boundary only.*

LEMMA 2. *If $f(m, n)$ is preharmonic everywhere and satisfies the inequality*[1]

$$|f(m, n)| < A\{1 + (|m| + |n|)^k\}$$

for all m, n, where k is a positive integer, then $f(m, n)$ is a polynomial of degree not exceeding k.

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[1] In what follows, A will always represent a positive nonzero number, independent of the variables in the context.

These lemmas are special cases of Theorems 1 and 6 of Heilbronn's paper.

LEMMA 3. *The function*

$$h(m, n) = \frac{1}{\pi}\int_0^\pi \cos mt\, \phi^n(t)dt,$$

where $\phi(t)$ is the smaller root of the equation

$$\phi(t) + \phi^{-1}(t) + 2\cos t = 4,$$

is preharmonic everywhere with the following properties:

(a) $h(0, 0) = 1$,

(b) $h(m, 0) = 0$ *for* $m \neq 0$,

(c) $h(m, n) > 0$ *for* $n > 0$,

(d) $|h(m, n) - n/\pi(m^2+n^2)| < A/n(m^2+n^2)$ *for all m and positive n.*

(a) and (b) follow by inspection. To prove (c), let

$$M(n) = \operatorname*{glb}_{|m|<\infty} h(m, n)$$

for $n \geqq 0$. It is easily seen that $|\phi(t)| < 1$ for $0 < t \leqq \pi$ and so $M(n) \to 0$ as $n \to \infty$ and, from the difference equation for preharmonic functions, we have for $n \geqq 1$

$$2M(n) \geqq M(n + 1) + M(n - 1)$$

and, since $M(0) = 0$, the result follows.

It may be verified that $\phi(t)$ is a positive decreasing function of t in $(0, \pi)$ with derivatives of all orders there, that

$$\phi'(\pi) = 0, \qquad \lim_{t\to 0+} \phi'(t) = -1, \tag{1}$$

$$\phi(t) = 1 - t + t^2/2 - t^3/12 + O(t^4) \tag{2}$$

as $t \to 0+$, and that there exists a real number $\eta > 0$ such that

$$\phi(t) \leqq e^{-\eta t} \qquad \text{for } 0 \leqq t \leqq \pi. \tag{3}$$

Integrating by parts twice in the expression for $h(m,n)$ we have from (1) and the fact that $\sin m\pi = 0$

$$\pi h(m, n) = \frac{n}{m^2} - \frac{n}{m^2}\int_0^\pi \phi^{n-2}(t)\cos mt\left[(n - 1)\{\phi'(t)\}^2 + \phi(t)\phi''(t)\right]dt,$$

or, adding $\pi(n^2/m^2)h(m, n)$ to each side,

$$\pi\cdot\frac{m^2+n^2}{m^2}\cdot h(m,n) = \frac{n}{m^2} - \frac{n}{m^2}\int_0^\pi \phi^{n-2}(t)\psi(t)\cos mtdt,$$

where

$$\psi(t) = (n-1)\{\phi'(t)\}^2 + \phi(t)\phi''(t) - n\phi^2(t).$$

From the enunciated properties of $\phi(t)$ we may easily show that

$$|\psi(t)| < A(|n|t^2 + t)$$

for $0 < t \leqq \pi$. Thus, by (3), we have for $n \geqq 1$

$$\left|\pi\cdot\frac{m^2+n^2}{m^2}\cdot h(m,n) - \frac{n}{m^2}\right| < A\frac{n}{m^2}\int_0^\pi e^{-n\eta t}(nt^2+t)dt$$

$$< \frac{A}{nm^2},$$

and this completes the proof of Lemma 3.

3. THEOREM 1. *A necessary and sufficient condition for a function $f(m, n)$ to be non-negative and preharmonic for $n \geqq 0$ is that the numbers $f(m, 0)$ $\{m = 0, \pm 1, \pm 2, \cdots\}$ should be non-negative and satisfy*

$$\sum_{m=-\infty}^{\infty} \frac{f(m,0)}{1+m^2} < \infty$$

and that there should exist a non-negative number D for which

$$f(m,n) = Dn + \sum_{r=-\infty}^{\infty} f(r,0)h(m-r,n) \tag{4}$$

for $n \geqq 0$.

SUFFICIENCY. For any large positive N and $n > 0$ we have, from Lemma 3(d),

$$\sum_{r=-N}^{N} f(r,0)h(m-r,n) < A\sum_{r=-N}^{N}\frac{f(r,0)n}{(m-r)^2+n^2}$$

$$< AC(m,n)\sum_{r=-N}^{N}\frac{f(r,0)}{1+r^2}$$

where

$$C(m,n) = \operatorname*{lub}_{|r|<\infty} \frac{n(1+r^2)}{(m-r)^2+n^2}$$

and is finite for any fixed m, n. Thus the function defined by (4) is an absolutely convergent series of non-negative preharmonic functions and, hence, is itself non-negative and preharmonic for $n \geqq 0$.

NECESSITY. Let R be a positive integer and define

$$f_R(m, n) = f(m, n) - \sum_{r=-R}^{R} f(r, 0) h(m - r, n).$$

Evidently $f_R(m, n)$ is preharmonic in the half-plane $n \geqq 0$ and also

$$f_R(m, n) \geqq - \left\{ \max_{|r| \leqq R} h(m - r, n) \right\} \sum_{r=-R}^{R} f(r, 0).$$

From Lemma 3(d) the right-hand side has arbitrarily small modulus for all points (m, n) of the half-plane lying outside a sufficiently large circle with centre at $(0, 0)$. Since, by Lemma 1, a preharmonic function in a finite domain attains its minimum on the boundary and $f_R(m, n) = 0$ for $n = 0$, it follows that for $n \geqq 0$, $f_R(m, n) \geqq 0$. That is to say, for $n \geqq 0$ we have

$$f(m, n) \geqq \sum_{r=-R}^{R} f(r, 0) h(m - r, n),$$

and letting $R \to \infty$

$$f(m, n) \geqq \sum_{r=-\infty}^{\infty} f(r, 0) h(m - r, n). \tag{5}$$

Next, by Lemma 3(d), there exists a large positive integer N for which

$$h(m, N) > 1/(N^2 + m^2)$$

for all integers m. Thus we have, from (5),

$$\begin{aligned} f(0, N) &\geqq \sum_{r=-\infty}^{\infty} f(r, 0) h(-r, N) \\ &\geqq \sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{r^2 + N^2} \\ &\geqq \frac{1}{N^2} \sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{1 + r^2}. \end{aligned}$$

This proves that if $f(m, n)$ is non-negative and preharmonic for $n \geqq 0$,

$$\sum_{r=-\infty}^{\infty} \frac{f(r, 0)}{1 + r^2} < \infty.$$

If we write

$$f_\infty(m, n) = f(m, n) - \sum_{r=-\infty}^{\infty} f(r, 0)h(m - r, n),$$

it remains to show that $f_\infty(m, n) = Dn$ for some non-negative D. Now since the series $\sum_{r=-\infty}^{\infty} f(r, 0)h(m-r, n)$ is convergent and each term is non-negative and preharmonic for $n \geqq 0$, $f_\infty(m, n)$ also is non-negative and preharmonic for $n \geqq 0$, and, a fortiori, for any integral $t > 0$, $f_\infty(m, n+t)$ is non-negative and preharmonic for $n \geqq 0$. From what we have just proved above, we have

$$\sum_{r=-\infty}^{\infty} \frac{f_\infty(r, t)}{1 + r^2} < \infty$$

and, a fortiori, $f_\infty(m, t) < K_t(1+m^2)$, where K_t is finite for each integral t. Let us assume for the moment that we have shown that

$$f_\infty(m, n) < An^2(1 + m^2) < A[1 + (|m| + |n|)^4] \tag{6}$$

for $n > 0$. We may continue $f_\infty(m, n)$ uniquely throughout the entire plane by writing

$$f_\infty(m, -n) = -f_\infty(m, n) \tag{7}$$

for $n > 0$, and have

$$\begin{aligned} &\text{(i)} \quad f_\infty(m, n) \text{ preharmonic everywhere,} \\ &\text{(ii)} \quad f_\infty(m, n) < A[1 + (|m| + |n|)^4] \text{ everywhere,} \\ &\text{(iii)} \quad f_\infty(m, 0) = 0 \text{ for all } m, \\ &\text{(iv)} \quad \operatorname{sign} f_\infty(m, n) = \operatorname{sign} n \text{ for } n \neq 0. \end{aligned} \tag{8}$$

Applying Lemma 2 to $f_\infty(m, n)$ it follows from (8)(ii) that $f_\infty(m, n)$ is a polynomial of degree not exceeding 4. From (8)(iii), n must be a factor of $f_\infty(m, n)$; since $f_\infty(m, n)$ by (7) contains only odd powers of n we must have

$$f_\infty(m, n) = n\phi(m, n^2),$$

where $\phi(m, n^2)$ is a polynomial of degree not exceeding 3. Further, from (8)(iv), $\phi(m, n^2)$ is everywhere non-negative, and so of degree not exceeding 2. We have now shown that

$$f_\infty(m, n) = n(\alpha m^2 + \beta n^2 + \gamma m + \delta)$$

where α and β are non-negative. It may be verified, from the difference equation, that since $f_\infty(m, n)$ is preharmonic, then $\alpha + 3\beta = 0$. Thus $\alpha = \beta = 0$ and this implies that $\gamma = 0$ and $\delta \geqq 0$. This completes the proof that

$$f_\infty(m, n) = Dn$$

for some non-negative D.

It remains to prove (6). Consider the function

$$g(m, n, \bar{m}, 2\bar{n}) = \frac{1}{\bar{n}} \sum_{r=1}^{2\bar{n}} \sin \frac{r\pi(m - \bar{m} + \bar{n})}{2\bar{n}} \cdot \sin \frac{r\pi}{2} \cdot \frac{\sinh \alpha_r n}{\sinh 2\alpha_r \bar{n}}$$

where α_r is the positive root of the equation

$$\cosh \alpha_r + \cos \frac{r\pi}{2\bar{n}} = 2. \tag{9}$$

This function is preharmonic everywhere[2] and may be shown to satisfy

$$g(m, n, \bar{m}, 2\bar{n}) = \begin{cases} 0 & \text{for} \quad m = \bar{m} \pm \bar{n}, \\ 0 & \text{for} \quad n = 0, \\ 0 & \text{for} \quad 1 \leqq |m - \bar{m}| \leqq \bar{n}, \ n = 2\bar{n}, \\ 1 & \text{for} \quad m = \bar{m}, \ n = 2\bar{n}. \end{cases}$$

Further,

$$g(\bar{m}, 1, \bar{m}, 2\bar{n}) = \frac{1}{\bar{n}} \sum_{r=1}^{2\bar{n}} \sin^2 \frac{r\pi}{2} \cdot \frac{\sinh \alpha_r}{\sinh 2\alpha_r \bar{n}}$$
$$\geqq \frac{1}{\bar{n}} \frac{\sinh \alpha_1}{\sinh 2\alpha_1 \bar{n}}.$$

From (9) we have

$$\cosh \alpha_r = 2 - \cos \frac{r\pi}{2\bar{n}} < \cosh \frac{r\pi}{2\bar{n}},$$

and so

$$\alpha_r < \frac{r\pi}{2\bar{n}}, \tag{10}$$

and substituting this in the inequality for $g(\bar{m}, 1, \bar{m}, 2\bar{n})$ we deduce that

$$g(\bar{m}, 1, \bar{m}, 2\bar{n}) > A/\bar{n}^2. \tag{11}$$

Let us suppose that (6) is not true. Then there exists an increasing sequence of integers $\{n_\nu\}$, and a corresponding sequence of integers $\{m_\nu\}$ such that as $\nu \to \infty$

[2] This method of writing preharmonic functions as a sum of products is due to Phillips and Wiener [5].

$$\frac{f_\infty(m_\nu, n_\nu)}{n_\nu^2(1 + m_\nu^2)} \to \infty.$$

We shall suppose first that the integers n_ν are even. Consider the function

$$\bar{f}_\nu(m, n) = f_\infty(m, n) - f_\infty(m_\nu, n_\nu)g(m, n, m_\nu, n_\nu).$$

From (10) and (11) it is apparent that $\bar{f}_\nu(m, n) \geqq 0$ on the boundary of the square $|m - m_\nu| \leqq n_\nu$, $0 \leqq n \leqq n_\nu$, and also, by Lemma 1, inside the square. In particular, we have

$$f_\infty(m_\nu, 1) \geqq f_\infty(m_\nu, n_\nu)g(m_\nu, 1, m_\nu, n_\nu),$$

and so, by (11),

$$f_\infty(m_\nu, n_\nu) < An_\nu^2(1 + m_\nu^2),$$

which contradicts our assumption. Similarly, if the sequence is odd, we may show that

$$f_\infty(m_\nu, n_\nu) < An_\nu^2(1 + m_\nu^2).$$

COROLLARY. *Suppose $f(m, n)$ to be preharmonic and non-negative in $n \geqq 0$. Then, as $n \to \infty$ subject to the condition $am + bn = 0$ where a and b are integers,*

$$f(m, n) - H(m, n) = Dn + O\{(m^2 + n^2)^{-1/2}\},$$

where D is a non-negative number and

$$H(m, n) = \frac{1}{\pi} \sum_{r=-\infty}^{\infty} f(r, 0) \cdot \frac{n}{(m - r)^2 + n^2} \cdot$$

This result follows immediately from Theorem 1 and Lemma 3(d).

4. If $H(re^{i\theta})$ is positive and harmonic in the half-plane $0 < \theta < \pi$, then it may be written as [2]

$$(12) \qquad H(re^{i\theta}) = dr \sin \theta + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{r \sin \theta}{r^2 - 2rt \cos \theta + t^2} \cdot dg(t)$$

where d is a non-negative number and $g(t)$ is a nondecreasing function such that

$$\int_{-\infty}^{\infty} \frac{dg(t)}{1 + t^2} < \infty.$$

LEMMA 4. *If $H(re^{i\theta})$ is defined by (12), $-1 < \rho < 1$, $0 < \phi < \pi$, n is an integer and α, δ are any positive numbers such that*

$$H(n\delta e^{i\phi}) \sim \alpha(n\delta)^\rho$$

as $n \to \infty$, *then*

$$H(re^{i\phi}) \sim \alpha r^\rho$$

as $r \to \infty$.

There is no loss in generality in assuming $d=0$, and so as $n \to \infty$

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dg(t)}{(n\delta)^2 - 2n\delta t \cos\phi + t^2} \sim \alpha \operatorname{cosec}\phi \cdot (n\delta)^{\rho-1}.$$

This is easily shown to imply that for $x > 1$

$$\frac{g(x) - g(-x)}{x^2} + \int_x^\infty \frac{d\{g(t) - g(-t)\}}{x^2 + t^2} < Ax^{\rho-1}.$$

Further, it will be sufficient to prove that for $|r-\sigma| \leqq 1$ and $r \to \infty$

$$H(re^{i\phi}) - H(\sigma e^{i\phi}) = o(r^\rho).$$

Now from (12), since $d=0$,

$$\begin{aligned}
&| H(re^{i\phi}) - H(\sigma e^{i\phi}) | \\
&\quad = \left| \frac{(r-\sigma)\sin\phi}{\pi} \int_{-\infty}^{\infty} \frac{(t^2 - r\sigma)dg(t)}{(r^2 - 2rt\cos\phi + t^2)(\sigma^2 - 2\sigma t\cos\phi + t^2)} \right| \\
&\quad \leqq A\left[\frac{g(r) - g(-r)}{r^2} + \int_r^\infty \frac{d\{g(t) - g(-t)\}}{r^2 + t^2}\right] \\
&\quad \leqq Ar^{\rho-1} \\
&\quad = o(r^\rho)
\end{aligned}$$

as $r \to \infty$.

The following two lemmas are contained in a paper by Allen and Kerr [4]:[3]

LEMMA 5. *If* $H(re^{i\theta})$ *is defined by* (12), $-1 < \rho < 1$, *and*

$$H(re^{i\pi/2}) \sim (1+\rho)\alpha r^\rho \sec \rho\pi/2$$

as $r \to \infty$, *then*

$$g(x) - g(-x) \sim 2\alpha x^{1+\rho}$$

as $x \to \infty$.

[3] Actually Allen and Kerr state their results for the case $r \to 0+$, but the case $r \to \infty$ is an elementary corollary.

LEMMA 6. *If $H(re^{i\theta})$ is defined by* (12), $-1<\rho<1$, *and*

$$H(re^{i\theta}) \sim (1+\rho) \operatorname{cosec} \rho\pi[\alpha \sin \rho(\pi-\theta)+\beta \sin \rho\theta]r^{\rho}, \tag{13}$$

as $r\to\infty$ for two distinct values of θ, then (13) *remains true for all values of θ and as $x\to\infty$*

$$g(x)-g(0) \sim \alpha x^{1+\rho}, \qquad g(0)-g(-x) \sim \beta x^{1+\rho}.$$

THEOREM 2. *If $f(m, n)$ is non-negative and preharmonic in the half-plane $n\geqq 0$, $-1<\rho<1$, and*

$$f(0, n) \sim (1+\rho)\alpha \sec \rho\pi/2\cdot n^{\rho}$$

as $n\to\infty$, then

$$\sum_{m=-R}^{R} f(m, 0) \sim 2\alpha R^{1+\rho}$$

as $R\to\infty$.

THEOREM 3. *If $f(m, n)$ is non-negative and preharmonic in the half-plane $n\geqq 0$, $-1<\rho<1$, and*

$$\begin{aligned} f(m, n) \sim (1+\rho) \operatorname{cosec} \rho\pi\Bigg[\alpha \sin \rho\left(\pi-\arctan\frac{n}{m}\right) \\ +\beta \sin \rho\left(\arctan\frac{n}{m}\right)\Bigg](n^2+m^2)^{\rho/2} \end{aligned} \tag{14}$$

as $n\to\infty$ for two distinct rational values of n/m, then (14) *remains true for all rational values of n/m, and as $R\to\infty$ we have*

$$\sum_{m=0}^{R} f(m, 0) \sim \alpha R^{1+\rho}, \qquad \sum_{m=-R}^{0} f(m, 0) \sim \beta R^{1+\rho}.$$

In (12) we define $g(x)$ to be constant in the interval $n<x<n+1$, for all integers n and with saltus $f(n, 0)$ at $x=n$: then Theorems 2 and 3 follow directly from the corollary to Theorem 1 and Lemmas 4, 5, and 6.

THEOREM 4. *If $f(m, n)$ is non-negative and preharmonic in the half-plane $n\geqq 0$, and for some finite $\bar{n}\geqq 0$*

$$\lim_{m\to\infty} f(m, \bar{n}) = \alpha,$$

then we have

$$\lim_{m\to\infty} f(m, n) = \alpha_n$$

for $n \geqq 0$ *where* α_n *is a linear function of* n.

$f(m, n)$ is non-negative and preharmonic in the half-plane $n \geqq \bar{n}$ and so by Theorem 1 has the representation

$$f(m, \bar{n} + p) = Dp + \sum_{r=-\infty}^{\infty} f(r, \bar{n}) h(m - r, p)$$

for $p > 0$, and some non-negative D. From the definition of $h(m, n)$ it is easily verified that

$$\sum_{m=-\infty}^{\infty} h(m, p) = 1. \tag{15}$$

Also, from Lemma 3(d), for $p > 0$

$$h(m, p) < Ap/(m^2 + p^2). \tag{16}$$

From the hypothesis and Theorem 1, given $\epsilon > 0$ there exists an integer $N > 0$ for which

$$| f(m, \bar{n}) - \alpha | \leqq \epsilon \tag{17}$$

for $m > N$ and for which

$$\sum_{r=-\infty}^{-N} \frac{f(r, \bar{n})}{1 + r^2} \leqq \epsilon. \tag{18}$$

We may now apply (15)–(18) as follows:

$$\begin{aligned}
| f(m, \bar{n} + p) - Dp - \alpha | \\
&\leqq \sum_{r=-\infty}^{\infty} | f(r, \bar{n}) - \alpha | \, h(m - r, p) \\
&\leqq \sum_{r=-\infty}^{N} f(r, \bar{n}) h(m - r, p) + \alpha \sum_{r=-\infty}^{N} h(m - r, p) \\
&\quad + \sum_{r=N+1}^{\infty} | f(r, \bar{n}) - \alpha | \, h(m - r, p) \\
&\leqq Ap \sum_{r=-\infty}^{-N} \frac{f(r, \bar{n})}{1 + r^2} + \frac{Ap}{(m - N)^2} \sum_{r=-N}^{N} f(r, \bar{n}) \\
&\quad + Ap\alpha \sum_{r=m-N}^{\infty} \frac{1}{r^2 + p^2} + \epsilon \sum_{r=N+1}^{\infty} h(m - r, p) \\
&\leqq Ap\epsilon + \frac{Ap}{(m - N)^2} \sum_{r=-N}^{N} f(r, \bar{n}) + \frac{Ap\alpha}{m - N} + \epsilon.
\end{aligned}$$

It is apparent that by a suitable choice of ϵ and correspondingly large m, the right-hand side is arbitrarily small. This proves the theorem for $n > \bar{n}$ and the complete result follows from the difference equation for preharmonic functions.

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A THEOREM OF PHRAGMÉN-LINDELÖF TYPE[1]

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1. **Introduction.** In the present paper the Phragmén-Lindelöf theorem for harmonic functions in the formulation of M. Heins [4] shall be extended to the solutions of the elliptic partial differential equation

$$L_k[u] \equiv \sum_1^n \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n}\frac{\partial u}{\partial x_n} = 0 \qquad (k < 1) \tag{1.1}$$

(k denoting a real constant). Equation (1.1) appears in several problems. For an exposition of previous results in the theory of the solutions of (1.1) we refer to a recent paper of A. Weinstein [9].

A theorem of Phragmén-Lindelöf type for the solutions of a rather general class of elliptic partial differential equations has been proved by D. Gilbarg [3] and E. Hopf [5]. Because of the singular coefficient, (1.1) is not contained in this class.

We introduce the following notations, $P(x_1, x_2, \cdots, x_n)$ denoting a point in the n-dimensional space:

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