CONCERNING CERTAIN TYPES OF WEBS

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In my dissertation[1] I defined a W_n set as follows: If $n>1$, a W_n set is a compact continuum M for which there exists a family F of n elements such that (1) each element of F is an upper semicontinuous collection of mutually exclusive continua which fills up M and is an arc with respect to its elements, and (2) if G is a collection of continua each belonging to some, but no two to the same, collection of the family F, then the continua of the collection G have a point in common and their common part is totally disconnected. In that paper it was stated[2] without proof that in the plane there exists a W_3 set M whose boundary is the point set $B(M)$ and which has a complementary domain whose boundary, J, contains six limit points of $B(M)-J$ but that no W_3 set has a complementary domain whose boundary contains seven such points. It is the purpose of this paper to prove this statement.

In what follows the space considered will be the plane and if M is a point set the notation $B(M)$ will be used to denote its boundary.

THEOREM 1. *There exists a W_3 set M whose outer boundary, J, contains six limit points of $B(M)-J$.*

PROOF. Let J denote a circle with interior E. Let $Q_1, Q_2, \cdots, Q_6$ be six points of J in that order. Let $\gamma_1, \gamma_2, \cdots, \gamma_6$ be six sequences of circles such that (1) for each i the sequence γ_i has a sequential limiting set which contains only the point Q_i, and (2) if x and y are two circles each belonging to one of the γ_i's then x is exterior to y and $x+y$ is a subset of E. Let D denote a point set such that P is a point of it if and only if there is a circle of one of the γ_i's whose interior contains P. Let M denote the point set $J+E-D$. We shall prove that M is a W_3 set.

Let C denote a circle which lies, together with its interior, I, in $M-B(M)$. There exist[3] three collections of arcs, H_1, H_2, and H_3, satisfying with respect to $C+I$ all the requirements of the definition of a W_3 set and such that (1) if h_i is an endelement of H_i, $h_i \cdot C$ is an

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[1] Mary-Elizabeth Hamstrom, *Concerning webs in the plane*, Trans. Amer. Math. Soc. vol. 74 (1953) pp. 500–513.

[2] Theorems 13 and 14.

[3] This follows from the argument used in the proof of Theorem 2 of my dissertation.

endpoint of h_i and (2) if h_i is a non-endelement of H_i, $h_i \cdot C$ consists of the endpoints of h_i. There is a reversible transformation of the number interval $[0, 1]$ into the collection H_1 such that if for each number x in this interval h_x denotes its image under this transformation and the number sequence $x_1, x_2, \cdots$ converges to the number a, then the limiting set of the sequence $h_{x_1}, h_{x_2}, \cdots$ is a subset of h_a. Let P_0 and P_1 denote $C \cdot h_0$ and $C \cdot h_1$ respectively. The circle C is the sum of two arcs, P_0XP_1 and $P_0X'P_1$. If $0<x<1$, h_x contains a point P_x on P_0XP_1 and a point P_x' on $P_0X'P_1$.

Let α_1 denote a positive number and let a and b be two numbers between 0 and 1 such that a is less than b. Let Q_1Q_2, Q_2Q_3, Q_3Q_4, Q_4Q_5, Q_5Q_6, and Q_6Q_1 be six nonoverlapping arcs whose sum is J. Let u_a be an arc lying except for its endpoints in $M-J$ such that $u_a \cdot J = Q_1 + Q_2$, $u_a \cdot (C+I) = h_a$, $u_a - h_a$ is the sum of a countable number of straight line intervals with slope α_1 or $-\alpha_1$ and the simple closed curve $u_a + Q_1Q_2$ encloses h_0 and every circle of γ_1 and γ_2 and neither encloses nor intersects any circle of the remaining γ_i's. Let u_b be an arc lying except for its endpoints in $M-(u_a+J)$ such that $u_b \cdot J = Q_4 + Q_5$, $u_b \cdot (C+I) = h_b$, $u_b - h_b$ is the sum of a countable number of straight line intervals with slope α_1 or $-\alpha_1$, and $u_b + Q_4Q_5$ encloses h_1 and every circle of γ_4 and γ_5 and neither encloses nor intersects any element of γ_3 or γ_6.

Let M_1, M_2, and M_3 denote $u_a + Q_1Q_2$ plus its interior, $Q_2Q_3Q_4 + Q_5Q_6Q_1 + u_a + u_b$ plus its interior, and $u_b + Q_4Q_5$ plus its interior respectively. There is a continuous collection, U_1, of mutually exclusive arcs and simple closed curves filling up $M \cdot M_1$ such that (1) U_1 is an arc with respect to its elements, (2) h_0 and $u_a + Q_1Q_2$ are the endelements of U_1, (3) if x is an element of γ_1 or γ_2 and u is an element of U_1 intersecting x then $u \cdot x$ is totally disconnected[4] and (4) if u is a non-endelement of U_1, $u \cdot (C+I)$ is an element of H_1 and $u - u \cdot (C+I)$ is the sum of a countable number of straight line intervals with slope α_1 or $-\alpha_1$.

There is a continuous collection, U_3, of mutually exclusive arcs and simple closed curves filling up $M_3 \cdot M$ such that (1) U_3 is an arc with respect to its elements, (2) h_1 and $u_b + Q_4Q_5$ are the endelements of U_3, (3) if x is an element of γ_4 or γ_5 and u is an element of U_3 intersecting x then $u \cdot x$ is totally disconnected, and (4) if u is a non-endelement of U_3, $u \cdot (C+I)$ is an element of H_1 and $u - u \cdot (C+I)$ is the sum of a

[4] It follows from Theorem 8 of my dissertation that if U is the collection of all elements of U_1 intersecting x, then the endelements of U are simple closed curves, the non-endelements of U are arcs, and no non-endelement intersects any element of γ_1 or γ_2 other than x.

countable number of straight line intervals with slope α_1 or $-\alpha_1$.

There is an upper semicontinuous collection, U_2, of mutually exclusive continuous curves filling up $M\cdot M_2$ such that (1) U_2 is an arc with respect to its elements, (2) u_a and u_b are the endelements of U_2, (3) if u is an element of U_2 it contains only one point of the arc $Q_2Q_3Q_4$ and only one point of the arc $Q_5Q_6Q_1$, the point set $u\cdot(C+I)$ is an element of H_1 and $u-u\cdot(C+I)$ is either the sum of a countable number of straight line intervals with slope α_1 or $-\alpha_1$ or the sum of some elements of γ_3 or γ_6 and a countable number of straight line intervals with slope α_1 or $-\alpha_1$, and (4) if u is an element of U_2 and it intersects an element of γ_3 or γ_6 it contains that element.

Let G_1 denote the sum of the collections U_1, U_2, and U_3. The collection G_1 is an upper semicontinuous collection of mutually exclusive continuous curves filling up M such that G_1 is an arc with respect to its elements and each element of G_1 is either (1) an element of H_1, (2) the sum of an element of H_1 and a countable number of straight line intervals with slope α_1 or $-\alpha_1$, or (3) the sum of an element of H_1, a countable number of straight line intervals with slope α_1 or $-\alpha_1$, and either Q_1Q_2, Q_4Q_5, or some elements of γ_3 or γ_6.

If, in the above, we replace α_1 by a positive number α_2 different from α_1 and if we replace H_1 by H_2, we can obtain an upper semicontinuous collection, G_2, of mutually exclusive continuous curves filling up M such that G_2 is an arc with respect to its elements and each element of G_2 is either (1) an element of H_2, (2) the sum of an element of H_2 and a countable number of straight line intervals with slope α_2 or $-\alpha_2$, or (3) the sum of an element of H_2, a countable number of straight line intervals with slope α_2 or $-\alpha_2$, and either Q_2Q_3, Q_5Q_6, or some elements of γ_4 or γ_1.

Replacing α_1 by a positive number α_3 different from α_1 and α_2 and replacing H_1 by H_3 we can obtain an upper semicontinuous collection, G_3, of mutually exclusive continuous curves filling up M such that G_3 is an arc with respect to its elements and each element of G_3 is either (1) an element of H_3, (2) the sum of an element of H_3 and a countable number of straight line intervals with slope α_3 or $-\alpha_3$, or (3) the sum of an element of H_3, a countable number of straight line intervals with slope α_3 or $-\alpha_3$, and either Q_3Q_4, Q_6Q_1 or some elements of γ_5 or γ_2.

The collections G_1, G_2, and G_3 satisfy with respect to M all the requirements of the definition of a W_3 set.

THEOREM 2. *If M is a W_3 set and J is the boundary of a complementary domain of M, then J does not contain seven limit points of $B(M)-J$.*

PROOF. The continuum J is a simple closed curve.[5] Suppose J does contain seven limit points of $B(M)-J$. Let α denote a collection of seven such points of J. The simple closed curve J is the sum of seven nonoverlapping arcs the sum of whose endpoints is the sum of the elements of α. Let β denote the collection of these arcs.

Let G_1, G_2, and G_3 be collections satisfying with respect to M all the requirements of the definition of W_3 set. For each i less than 4 let H_i denote the collection of all elements of G_i which intersect J. We shall prove first that each of these collections consists of more than one element.

Suppose H_3 consists of only one element, h. Since J is a subset of h and M is a W_3 set with respect to G_1, G_2, and G_3, each of the collections H_1 and H_2 contains more than one element. Consequently, for each i less than 3, H_i contains[6] two elements, h_i and h_i', neither of which separates H_i^*.[7] Let U_i denote $H_i^*-(h_i+h_i')$. Since $h_i\cdot J$ and $h_i'\cdot J$ are connected and $h_i\cdot h$ and $h_i'\cdot h$ are totally disconnected, $h_i\cdot J$ and $h_i'\cdot J$ are degenerate. Consequently U_1 and U_2 each contains five points of α and one of these points is in $U_1\cdot U_2$. Let P denote one such point.

Since U_1 and U_2 are open subsets of M and P is a limit point of $B(M)-J$, there is a complementary domain of M whose boundary, J', intersects $U_1\cdot U_2$. The continuum J' is therefore a subset[8] of an element of H_1 and an element of H_2, which is contrary to the assumption that M is a W_3 set with respect to G_1, G_2, and G_3. Hence H_3 contains more than one element. Let h_3 and h_3' be the elements of H_3 which do not separate H_3^* and let U_3 denote $H_3^*-(h_3+h_3')$.

For each i less than 4 let k_i denote the number of points of α lying in U_i. There are $7-k_i$ points of α in h_i+h_i' and if k_i is not greater than 4 then each of at least $7-k_i-2$ arcs of β is a subset of h_i or h_i'. Let l_i denote $5-k_i$. If k_i exceeds 4 let l_i be 0. In any case, $5-k_i\leqq l_i$. Since no point of α is common[9] to two of the point sets U_1, U_2, and U_3, $k_1+k_2+k_3\leqq 7$. Since no arc of β is a subset of two elements of the sum of the collections H_1, H_2, and H_3, $l_1+l_2+l_3\leqq 7$. However, $l_1+l_2+l_3\geqq +(5-k_1)+(5-k_2)+(5-k_3)\geqq 8$. Thus the assumption that J contains seven limit points of $B(M)-J$ leads to a contradiction.

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[5] R. H. Bing, *Concerning simple plane webs*, Trans. Amer. Math. Soc. vol. 60 (1946) pp. 133–148, Theorem 2.

[6] See Theorem 8 of my dissertation.

[7] If H is a collection of point sets, the notation H^* is used to denote the sum of the elements of H.

[8] See the corollary to Theorem 8 of my dissertation.

[9] This follows from an argument similar to that used to prove that H_3 does not contain only one element.

In a great many cases the methods used in the proofs of the above theorems can be used to determine whether a given continuum is a W_n set. In particular, they can be used to prove that no W_7 set, M, has a complementary domain whose boundary, J, contains three limit points of $B(M)-J$, no W_4 set has a complementary domain whose boundary contains five such points, and that there exists a W_6 set whose outer boundary contains three such points.

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ANOTHER REMARK ON "SOME PROBLEMS IN CONFORMAL MAPPING"

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It was remarked in [2] and proved in [3] that for every triply-connected domain D there are certain triply-connected subdomains D' having the same topological situation and admitting no conformal mapping into D preserving this topological situation other than the identity. This result implies at once several others. Indeed let D have contours K_1, K_2, K_3 and let the corresponding contours of D' be K_1', K_2', K_3'. It is assumed no contour of D reduces to a point. If D' is obtained from D by producing slits from K_2, K_3 out onto the same connected piece of the line of symmetry of D, it is clear that there is no conformal mapping of D' into D which can make K_2' go into K_2 or K_3' go into K_3 (in the natural sense of boundary correspondence). Thus for a domain D and subdomain D' there may exist no conformal mapping of the above type which carries either (a) a given boundary contour of D' into the corresponding boundary contour of D or (b) some two boundary contours of D' into the corresponding two boundary contours of D.

The question naturally arises whether given a triply-connected domain D and a triply-connected subdomain D' having the same topological situation there exists a conformal mapping of D' into D preserving the topological situation and carrying some one contour of D' into the corresponding contour of D. This question was raised to me by Professor A. Beurling some four or five years ago. The simple example above is not sufficient to provide an answer since in it

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