It is apparent that by a suitable choice of ϵ and correspondingly large m, the right-hand side is arbitrarily small. This proves the theorem for $n > \bar{n}$ and the complete result follows from the difference equation for preharmonic functions.

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A THEOREM OF PHRAGMÉN-LINDELÖF TYPE[1]

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1. **Introduction.** In the present paper the Phragmén-Lindelöf theorem for harmonic functions in the formulation of M. Heins [4] shall be extended to the solutions of the elliptic partial differential equation

$$L_k[u] \equiv \sum_{1}^{n} \frac{\partial^2 u}{\partial x_i^2} + \frac{k}{x_n} \frac{\partial u}{\partial x_n} = 0 \qquad (k < 1) \tag{1.1}$$

(k denoting a real constant). Equation (1.1) appears in several problems. For an exposition of previous results in the theory of the solutions of (1.1) we refer to a recent paper of A. Weinstein [9].

A theorem of Phragmén-Lindelöf type for the solutions of a rather general class of elliptic partial differential equations has been proved by D. Gilbarg [3] and E. Hopf [5]. Because of the singular coefficient, (1.1) is not contained in this class.

We introduce the following notations, $P(x_1, x_2, \cdots, x_n)$ denoting a point in the n-dimensional space:

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$$H = E[P \mid x_n > 0], \qquad H_r = H \cap E\left[P \,\middle|\, \sum_1^n x_i^2 < r^2\right],$$

$$D = E[P \mid x_n = 0], \qquad D_r = D \cap E\left[P \,\middle|\, \sum_1^n x_i^2 < r^2\right],$$

$$S = H \cap E\left[P \,\middle|\, \sum_1^n x_i^2 = 1\right], \quad S_r = H \cap E\left[P \,\middle|\, \sum_1^n x_i^2 = r^2\right].$$

$Q(\xi_1, \xi_2, \cdots, \xi_n)$ shall exclusively be used for a point on S_r, $Q^*(\xi_1/r, \xi_2/r, \cdots, \xi_n/r)$ for its radial projection on S. ϕ shall denote the angle between OQ (or OQ^*) and the hyperplane D; it is defined by the relation $\sin \phi = \xi_n/r$.

2. **Theorem.** *Let u be a solution of $L_k[u]=0$ $(k<1)$, defined in H and satisfying at the boundary*

$$\limsup_{P \to M} u(P) \leqq 0 \qquad (P \in H;\ M \in D). \tag{2.1}$$

It follows that

(a) *the limit $\alpha = \lim_{r\to\infty} m(r)/r^{1-k}$, where $m(r) = \sup_{P \in S_r} u(P)$, always exists (finite or infinite),*

(b) $\alpha \geqq 0$,

(c) $u \leqq \alpha x_n^{1-k}$ *holds throughout H,*

(d) *if in* (c) *the equality is attained in at least one point of H, then* $u \equiv \alpha x_n^{1-k}$.

Remarks. This theorem has first been proved for the case of harmonic functions in the plane by M. Heins [**4**], who thus had solved a question raised by L. Ahlfors [**1**]. (The three-dimensional case has later been treated in an analogous way by H. Keller [**7**].)

Although the result is well known, the following proof might also be of interest in the particular case of harmonic functions $(k=0)$, since it differs in some respects from the one given by M. Heins.

For $k \geqq 1$ a theorem of this type cannot be expected to hold, as the examples $u = \log x_n$ $(k=1)$ and $u = -x_n^{1-k} + \text{const.}$ $(k>1)$ show.

3. **Proof.** Let $f(Q)$ be defined and continuous on the boundary of H_r, vanishing on D_r. It has been shown in [**6**] that the function

$$\begin{aligned} v(P) = {} & x_n^{1-k} \frac{(r^2 - \rho^2)\Gamma((2-k+n)/2)}{r\pi^{n/2}\Gamma((2-k)/2)} \int_{S_r} \xi_n f(Q)\, dS_r(Q) \\ & \cdot \int_0^\pi \frac{\sin^{1-k} t\,dt}{[[PQ]^2 + 2x_n\xi_n(1-\cos t)]^{(2-k+n)/2}}, \end{aligned} \tag{3.1}$$

where $\rho^2 = \sum_1^n x_i^2$ and $[PQ]^2 = \sum_1^n (x_i - \xi_i)^2$, is the solution of (1.1) in H_r, which assumes the boundary values $f(Q)$.

Now, let $f(Q)$ be an arbitrary majorant of u on S_r. From the maximum principle we conclude that $u(P)$ is majorized by $v(P)$ in H_r. For fixed P, as Q tends to infinity, we have

$$[PQ]^2 + 2x_n\xi_n(1 - \cos t) = O(r^2). \tag{3.2}$$

Therefore, if majorants $f_i(Q)$ are given for a sequence of radii $r_i \to \infty$, we get, considering a fixed point P,

$$u(P) \leqq x_n^{1-k}[C + o(1)]r_i^{-n+k-1} \int_{S_r} \xi_n f_i(Q) dS_{r_i}(Q), \tag{3.3}$$

where

$$C = \frac{\Gamma((2 - k + n)/2)}{\pi^{(n-1)/2}\Gamma((3 - k)/2)} \cdot$$

(For another deduction of (3.3) in the case of harmonic functions in the plane see L. Bieberbach [2, pp. 132–134].) We define[3]

$$\alpha = \liminf_{r\to\infty} \frac{m(r)}{r^{1-k}} \cdot \tag{3.4}$$

There exists a sequence $r_i \to \infty$ such that $(\alpha + \epsilon_i)r_i^{1-k}$ is a majorant of u on S_{r_i}, where $\epsilon_i \to 0$ if $i \to \infty$. (In the case $\alpha = -\infty$ we have to replace $(\alpha + \epsilon_i)$ by δ_i, $\delta_i \to -\infty$.) If we apply (3.3) to the sequence $\{S_{r_i}\}$ while considering a fixed point $P \in H$, we obtain in the limit

$$u(P) \leqq x_n^{1-k} \alpha C \int_S \sin \phi dS = \alpha_1 x_n^{1-k}, \tag{3.5}$$

where

$$\alpha_1 = \alpha \frac{2\Gamma((2 - k + n)/2)}{(n - 1)\Gamma((n - 1)/2)\Gamma((3 - k)/2)} \cdot$$

(In the case of harmonic functions in the plane we have $\alpha_1 = 4\alpha/\pi$, a relation due to R. Nevanlinna [8, p. 43].)

By means of (3.5) the cases $\alpha = 0$ and $\alpha = -\infty$ can be easily treated. From now on we require α to be finite and different from zero.

It can be verified by computation that

$$C \int_S \sin^{2-k} \phi dS = 1. \tag{3.6}$$

[3] The trivial case $\alpha = +\infty$ shall be omitted in the sequel.

Therefore, since

$$C\int_S \sin \phi dS > C\int_S \sin^{2-k} \phi dS = 1,$$

we infer from (3.5) that

$$\alpha_1 \gtrless \alpha \qquad \text{for } \alpha \gtrless 0. \tag{3.7}$$

On the half-sphere S_{r_i}, u is majorized by the two functions $(\alpha+\epsilon_i)r_i^{1-k}$ and $\alpha_1(r_i \sin \phi)^{1-k}$. Putting either of them into (3.3), we would obtain (3.5) again. But we can improve (3.5) by combining the two majorants:

We define $h_1(Q^*) = \min\ [\alpha_1 \sin^{1-k} \phi, \alpha]$. Clearly u is majorized on S_{r_i} by $(h_1(Q^*)+\epsilon_i)r_i^{1-k}$. Hence (3.3) yields in the limit

$$u(P) \leqq \alpha_2 x_n^{1-k}, \qquad \text{where } \alpha_2 = C\int_S h_1 \sin \phi dS.$$

We can repeat this argument and obtain the following iteration:

If we define

$$h_\mu(Q^*) = \min\ [\alpha_\mu \sin^{1-k} \phi, \alpha] \qquad (\mu = 1, 2, 3, \cdots) \tag{3.8}$$

and

$$\alpha_\mu = C\int_S h_{\mu-1} \sin \phi dS \qquad (\mu = 2, 3, 4, \cdots), \tag{3.9}$$

we will have

$$u(P) \leqq \alpha_\mu x_n^{1-k} \qquad (\mu = 1, 2, 3, \cdots). \tag{3.10}$$

From (3.6), (3.8), and (3.9) we infer

$$\alpha_{\mu+1} = C\int_S h_\mu \sin \phi dS < C\alpha_\mu \int_S \sin^{2-k} \phi dS = \alpha_\mu, \tag{3.11}$$

i.e. the sequence $\{\alpha_\mu\}$ decreases monotonously. We have

$$\alpha_\mu \geqq \alpha \qquad \text{for } \alpha > 0, \tag{3.12}$$

because $\alpha_\mu < \alpha$ and (3.10) would lead to a contradiction with (3.4).

We shall now prove the relations (μ tending to infinity)

$$\alpha_\mu \searrow \alpha \qquad \text{for } \alpha > 0 \tag{3.13}$$

and

$$\alpha_\mu \searrow -\infty \qquad \text{for } \alpha < 0. \tag{3.14}$$

The proof of (a), (b), and (c) will thus be achieved, because these statements are immediate consequences from (3.10), (3.13), and (3.14).

Proof of (3.13). If (3.13) were not true, we could conclude from (3.11) and (3.12) that $\alpha_\mu \searrow \gamma$, where $\gamma > \alpha$. Using (3.6), (3.8), and (3.9) we would get

$$\begin{aligned}\alpha_\mu - \alpha_{\mu+1} &= C\int_S (\alpha_\mu \sin^{1-k}\phi - h_\mu)\sin\phi dS \\ &= C\int_{S^\mu} (\alpha_\mu \sin^{1-k}\phi - \alpha)\sin\phi dS \\ &> C\int_{S^\gamma} (\gamma \sin^{1-k}\phi - \alpha)\sin\phi dS > 0,\end{aligned}$$

where

$$S^\mu = E[Q^* \mid \alpha_\mu \sin^{1-k}\phi > \alpha] \quad \text{and} \quad S^\gamma = E[Q^* \mid \gamma \sin^{1-k}\phi > \alpha].$$

But since the last integral is independent of μ, we thus get a contradiction to the convergence of $\{\alpha_\mu\}$.

Proof of (3.14). If (3.14) were not true, we could conclude from (3.7) and (3.11) that $\{\alpha_\mu\}$ converges; more exactly, that $\alpha_\mu \searrow \delta$ where $-\infty < \delta < \alpha$. Making use of (3.6), (3.8), and (3.9) we would obtain

$$\begin{aligned}\alpha_\mu - \alpha_{\mu+1} &= C\int_S (\alpha_\mu \sin^{1-k}\phi - h_\mu)\sin\phi dS \\ &= C\int_{S^\mu} (\alpha_\mu \sin^{1-k}\phi - \alpha)\sin\phi dS \\ &> C\int_{S^\delta} (\delta \sin^{1-k}\phi - \alpha)\sin\phi dS > 0,\end{aligned}$$

where

$$S^\mu = E[Q^* \mid \alpha_\mu \sin^{1-k}\phi > \alpha] \quad \text{and} \quad S^\delta = E[Q^* \mid \delta \sin^{1-k}\phi > \alpha].$$

But since the last integral is independent of μ, we have a contradiction to the convergence of $\{\alpha_\mu\}$. Because of the maximum principle the function $\alpha x_n^{1-k} - u(P)$ cannot attain its maximum 0 in an interior point of H, unless it vanishes identically. This proves (d).

4. Final remarks. The theorem may also be applied to functions which possess continuous second derivatives and satisfy the differential inequality $L_k[u] \leqq 0$ $(k<1)$. More generally, it is true for a set of functions which for $k=0$ coincides with the class of subharmonic functions and which can be defined in analogous ways.

The theorem remains valid if H is replaced by any infinite subregion.

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