A DERIVATIVE TEST FOR FINITE SOLUTIONS OF GAMES

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For a continuous real-valued function M (called the payoff[1]) on the unit square, the minimax theorem of game theory states that there is a unique constant v (the value) and probability distributions f, g over $[0, 1]$ (called maximizing and minimizing optimal strategies, respectively) for which

$$(1) \qquad \int_0^1 M(x, y)df(x) \geqq v \geqq \int_0^1 M(x, y)dg(y).$$

At present there is no general method for finding these optimal strategies (which need not be unique), and, consequently, results which reduce the set of distributions among which one need search are desirable. Recently Samuel Karlin [3] has introduced a class of payoff functions M, related to convex payoff functions [1], for which optimal strategies with at most a finite number of points of increase always exist. The purpose of this note is to give a proof for all n of the theorem concerning these functions which Karlin proved for $n \leqq 4$:

THEOREM. *If $\partial^n M/\partial y^n$ is continuous and is of one sign, then there is a maximizing optimal strategy with at most n points of increase and a minimizing optimal strategy with at most $n/2$ points of increase (counting, in this case, 0 and 1 as half points).*

PROOF. As is shown in [3], we need only consider the case in which the partial never vanishes, and clearly we may take $v = 0$. Let f_0 be a fixed maximizing optimal strategy. From (1) it is apparent that any optimal g places weight at only the minima of $k(y) = \int_0^1 M(x, y)df_0(x)$; since $k^{(n)}(y) = \int_0^1 (\partial^n M/\partial y^n)df_0(x) \neq 0$, a simple argument based on Rolle's Theorem shows there are at most $n/2$ minimum points, counting 0 and 1 as half points, and the second assertion of the theorem follows. Furthermore, since $k^{(n)}(y) \neq 0$, it follows from Rolle's Theorem that the total number of roots of k, counting multiplicities, is at most n.

To establish the first assertion of the theorem we shall construct an auxiliary polynomial-like payoff function [2], the construction being based on this last observation. We first assert that there is a

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[1] For game-theoretic notation and motivation the reader is referred to [4, preface] and [5].

polynomial p_0 of degree less than n for which, on $[0, 1]$, $k \geq p_0 \geq 0$ and $k - p_0$ has exactly n roots counting multiplicities. For consider the nonvoid set of polynomials p of degree less than n for which $k \geq p \geq 0$. Counting multiplicities, the number of roots of $k - p$ is at most n (since $k^{(n)} - p^{(n)} = k^{(n)} \neq 0$) for these polynomials, so let p_0 be one for which the number of roots is maximum. If $k - p_0$ has roots y_i of multiplicity m_i, $i = 1, \cdots, r$, and $\sum m_i < n$, then, since the roots y_i interior to $[0, 1]$ are of even multiplicity, there is a polynomial $p_1 \geq 0$ (on $[0, 1]$) of degree less than n having just these roots and multiplicities. Consequently we may define the continuous function $(k - p_0)/p_1$ by setting

$$((k - p_0)/p_1)(y_i) = (k_0^{(m_i)}(y_i) - p_0^{(m_i)}(y_i))/p_1^{(m_i)}(y_i)$$

and clearly $\lambda = \min_y ((k - p_0)/p_1)(y) > 0$. But then $k \geq p_0 + \lambda p_1 \geq 0$, and, since $k - p_0 - \lambda p_1$ can thus have at most $\sum m_i$ roots, counting multiplicities, we must have $k^{(m_i)}(y_i) - p_0^{(m_i)}(y_i) - \lambda p_1^{(m_i)}(y_i) > 0$ and $k(y) - p_0(y) - \lambda p_1(y) > 0$ for $y \neq y_i$, $i = 1, \cdots, r$, so that $((k - p_0)/p_1)(y) > \lambda = \min (k - p_0)/p_1$, a contradiction. Thus we must have $\sum m_i = n$, so that p_0 is the desired polynomial.

Now it is easily seen that for each fixed x one may find coefficients $a_j(x)$, $j = 0, \cdots, n-1$, for which $\phi_x(y) = M(x, y) - \sum_{j=0}^{n-1} a_j(x) y^j$ has the roots y_i of multiplicity m_i. Moreover, this polynomial in y is unique,[2] so that, since we may write n linear equations for the unknowns $a_0(x), \cdots, a_{n-1}(x)$, the matrix of coefficients in these equations is nonsingular, and the a_j are seen to be continuous functions.

For any x, ϕ_x has a nonvanishing nth derivative, so that its only roots are the y_i (of multiplicity m_i). Since those interior to $[0, 1]$ are of even multiplicity, ϕ_x is of one sign and not identically zero—thus by continuity either

$$(2) \qquad\qquad M(x, y) - \sum_{j=0}^{n-1} a_j(x) y^j \geq 0 \qquad\qquad \text{for all } x, y$$

or

$$(3) \qquad\qquad M(x, y) - \sum_{j=0}^{n-1} a_j(x) y^j \leq 0 \qquad\qquad \text{for all } x, y.$$

But integration of this expression with respect to f_0 yields $k(y) - \sum_{j=0}^{n-1} (\int_0^1 a_j(x) df_0(x)) y^j$, and since this function of y has roots y_i

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[2] The difference of two such polynomials, of degree less than n, has n roots, hence $= 0$.

of multiplicity m_i (since this was true for each ϕ_x), we must have $p_0(y) - \sum_{j=0}^{n-1} (\int_0^1 a_j(x)df_0(x))y^j = 0$, as a polynomial of degree less than n with n roots. Since $k(y) > p_0(y)$ for some y, (2) must hold.

Now if g_0 is any minimizing optimal strategy, we have

$$\int_0^1 \sum a_j(x)y^j dg_0(y) \leqq \int_0^1 M(x, y)dg_0(y) \leqq 0 \leqq p_0(y)$$

$$= \int_0^1 \sum a_j(x)y^j df_0(x)$$

so that the payoff function $\sum_{j=0}^{n-1} a_j(x)y^j$ has value $v = 0$ (cf. (1)). In view of its polynomial-like character [2] it has a maximizing optimal strategy f with at most n points of increase, so that

$$\int_0^1 M(x, y)df(x) \geqq \int_0^1 \sum a_j(x)y^j df(x) \geqq 0,$$

and f is a maximizing optimal strategy for M, which completes the proof.

A second theorem of Karlin now appears as a

COROLLARY. *If $\partial^n M/\partial y^n$ is of one sign and never zero, and there is a k-dimensional set of minimizing optimal strategies, then there is a maximizing optimal strategy with at most $n - k$ points of increase.*

For since $M(x, y) \geqq \sum a_j(x)y^j$ and both functions are payoffs with value $v = 0$, if g is a minimizing optimal strategy for M, then $0 \geqq \int_0^1 \sum a_j(x)y^j dg(y)$, i.e., g is a minimizing optimal strategy for $\sum a_j(x)y^j$—thus this polynomial-like function has a k-dimensional set of minimizing optimal strategies, and, in view of the results of [2], has a maximizing optimal strategy with at most $n - k$ points of increase which, as before, is optimal for M.

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