

THE 1-FACTORS OF ORIENTED GRAPHS

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1. **Introduction.** A *graph* G consists of a set V of elements called *vertices* together with a set E of elements called *edges*, the two sets having no element in common. With each edge there are associated either one or two vertices called its *ends*.

In this paper we consider only graphs in which each edge has two distinct ends. We call such a graph *oriented* if one end of each edge is distinguished as the *positive* and the other as the *negative* end. If x is the positive end and y the negative end of an edge A we say that x is *joined to* y by A or that y is *joined from* x by A . We say also that A is a *join* from x to y .

An oriented graph G is *finite* if both V and E are finite, and is *infinite* otherwise.

If G is oriented and $x \in V$ we denote by $d'(x)$ and $d''(x)$ the cardinals of the sets of edges of G having x as positive and negative end respectively. We call G *simple* if $d'(x) = d''(x)$ for each $x \in V$. An oriented graph is *regular of degree* d if $d'(x) = d''(x) = d$ for each $x \in V$.

An oriented graph H is a *subgraph* of the oriented graph G if the vertices and edges of H are vertices and edges respectively of G and if each edge of H has the same positive end and the same negative end in H as in G .¹ If in addition each vertex of G is a vertex of H , then H is a *spanning subgraph* of G . If $S \subseteq V$ there is a subgraph of G whose vertices are the members of S and whose edges are those members of E which have both their ends in S . We denote this subgraph by $G[S]$. We find it convenient to write G_S for the graph $G[V-S]$.

If n is any positive integer we define an *n-factor* of an oriented graph G as a spanning subgraph of G which is regular of degree n . In this paper we obtain a necessary and sufficient condition that a given locally finite oriented graph shall have a 1-factor. By saying that G is locally finite we mean that $d'(x)$ and $d''(x)$ are finite for each $x \in V$. The theory is closely analogous to that for unoriented graphs given in [4] and [5].

2. **Paths and oriented paths.** Let G be any oriented graph.

A *path* in G is a sequence

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¹ For an unoriented graph G we define a *subgraph* H as a graph whose vertices and edges are vertices and edges respectively of G and in which each edge has the same ends as in G .

$$(2.1) \quad P = (v_0, e_1, v_1, e_2, \dots)$$

in which the terms are alternately vertices v_i of G and edges e_i of G , and which satisfies the condition that v_{i-1} and v_i are the two ends of e_i for each relevant i . It is not required that all the terms of P shall represent distinct edges or vertices of G . If they do we say that P is *simple*. The vertex v_0 is the *origin* of P .

The sequence P may be infinite. If not we postulate as part of the definition of a path that it terminates with a vertex v_n . We call v_n the *terminus* of P and say that P is a path *from* v_0 *to* v_n . The integer n is the *length* of the finite path. We admit the case in which P has only one term, v_0 . In this case we call P a *degenerate* path.

If the terms of a finite path P represent distinct edges and vertices of G except that the terminus is the same vertex as the origin, we say P is *circular*.

A path P represented by (2.1) is a *direct oriented path* in G if v_{i-1} is the positive and v_i the negative end of e_i , for each term e_i . It is a *reverse oriented path* if v_{i-1} is the negative and v_i the positive end of e_i , for each term e_i .

A subset S of V is *independent* in G if there is no nondegenerate finite direct oriented path whose origin and terminus are elements of S . If in addition there is no pair of infinite simple oriented paths in G , one direct and the other reverse, such that the origins of both paths belong to S , we say that S is *strictly independent* in G . For a finite G the terms *independent* and *strictly independent* are synonymous.

We shall prove the following

THEOREM. *A locally finite oriented graph G has no 1-factor if and only if there exist disjoint finite sets S and T of vertices of G such that T is strictly independent in G_S and $\alpha(S) < \alpha(T)$.*

(We denote the number of elements of a finite set U by $\alpha(U)$.)

We define connection in G by means of the unoriented paths, just as for an unoriented graph. Vertices x and y in G are *connected* in G if there is a finite path in G from x to y . The relation of being connected in G is easily seen to be an equivalence relation. It partitions the set of vertices of G into disjoint equivalence classes. If W is one of these equivalence classes we call $G[W]$ a *component* of G . The graph G is *connected* if it has only one component (or is *null*, that is has no edges and no vertices). Clearly the components of G are non-null connected graphs and two vertices of G are connected in G if and only if they are vertices of the same component of G .

If G is finite or locally finite there are only a finite number of paths

in G with a given origin and a given finite length n . As a consequence of this we have

(2.2) *In any connected locally finite graph the set of vertices and the set of edges are denumerable.*

3. 1-factors and determinants. In this section we at first take G to be a finite oriented graph. We suppose its vertices enumerated as a_1, a_2, \dots, a_m , if V is non-null. We then associate with G an $m \times m$ matrix $K(G) = \{k_{ij}\}$. We put $k_{ij} = 0$ if $i = j$ or if there is no edge of G having a_i as its positive end and a_j as its negative end. We regard the remaining elements k_{ij} of $K(G)$ as independent indeterminates over the field of rational numbers.

The determinant $|K(G)|$ is given by

$$(3.1) \quad |K(G)| = \sum \epsilon k_{1\alpha} k_{2\beta} \cdots k_{m\mu}$$

where $(\alpha\beta \cdots \mu)$ runs through the permutations of the integers 1 to m , and ϵ is $+1$ or -1 according as the permutation $(\alpha\beta \cdots \mu)$ is even or odd.

If $\epsilon k_{1\alpha} k_{2\beta} \cdots k_{m\mu}$ is a nonzero term of the sum (3.1), then for each k_{ij} in this product we can select an edge of G having a_i as positive end and a_j as negative end and so construct a 1-factor of G . Conversely for each 1-factor of G there is a nonzero term in the expansion of $K(G)$.

We call G *prime* if it has no 1-factor. We have shown that if G is non-null then

$$(3.2) \quad G \text{ is prime if and only if } |K(G)| = 0.$$

For a null graph N it is convenient to say that N is its own 1-factor and that $|K(N)| = 1$. Then (3.2) holds for all graphs.

Two distinct vertices a_i and a_j of G are *joined factorially from a_i to a_j* if there is a subgraph H of G for which $d'(a_i) = 0$, $d'(a_k) = 1$ if $k \neq i$, $d''(a_j) = 0$, and $d''(a_k) = 1$ if $k \neq j$. We call H a *factorial join from a_i to a_j* .

If K_{ij} denotes the cofactor of the element k_{ij} of $K(G)$, then in the case $i \neq j$ the factorial joins from a_i to a_j are associated with the nonzero terms in the expansion of K_{ij} just as the 1-factors of G are associated with the nonzero terms in the expansion of $|K(G)|$. Thus we have

(3.3) *Let a_i and a_j be distinct vertices of G . Then there is a factorial join from a_i to a_j in G if and only if $K_{ij} \neq 0$.*

By the theory of determinants the cofactors K_{ij} satisfy the equation

$$(3.4) \quad K_{pq} K_{rs} - K_{ps} K_{qr} = 0$$

whenever $|K(G)| = 0$, that is for any prime finite oriented graph G . (See, for example, [1, p. 98].)

A vertex a_i of G is *out-singular* if there is no factorial join from a_i to any other vertex of G , and *in-singular* if there is no factorial join from any other vertex of G to a_i . The following proposition is an immediate consequence of (3.2) and (3.4).

(3.5) *Let a_i and a_j be distinct vertices of G such that there is no factorial join from a_i to a_j . Then either a_i is out-singular or a_j is in-singular.*

(3.6) *Let G be a locally finite oriented graph having disjoint finite sets S and T of vertices such that T is strictly independent in G_S and $\alpha(T) > \alpha(S)$. Then G has no 1-factor.*

PROOF. Assume that G has a 1-factor F . Then if x is any element of T we can construct a direct oriented path $P(x)$ in G according to the following rules:

(i) $P(x)$ is nondegenerate and its first edge X_1 is the edge of F having x as its positive end.

(ii) If the construction has been carried as far as the i th edge X_i , we take the negative end x_i of X_i as the $(i+1)$ th vertex.

(iii) Suppose the construction has been carried as far as the $(i+1)$ th vertex x_i . Then we consider the construction completed if $x_i \in S$. Otherwise we take as $(i+1)$ th edge of $P(x)$ the edge of F having x_i as its positive end.

We note that $P(x)$ either has an element of S as terminus or is an infinite path in G_S . In the former case no term of $P(x)$ other than the last is an element of S .

Since T is independent in G_S no one of the vertices x_1, x_2, \dots is an element of T . It follows that $P(x)$ is a simple path. For if not, there are i, j such that $1 \leq i < j$; $x_i = x_j$; $X_i \neq X_j$. Then X_i and X_j are distinct edges of F with the same negative end, contrary to the definition of a 1-factor.

If x and y are distinct elements of T then $P(x)$ and $P(y)$ have no common vertex. For if they have let x_i be the first term of $P(x)$ which is a vertex of $P(y)$ and let y_j be one occurrence of that vertex in $P(y)$. Then the edges which immediately precede x_i and y_j in $P(x)$ and $P(y)$ respectively are distinct edges of F with the same negative end, contrary to the definition of a 1-factor.

Let n be the number of elements x of T such that $P(x)$ is infinite. The remaining $\alpha(T) - n$ elements x of T define finite paths $P(x)$ having $\alpha(T) - n$ distinct elements of S as termini. Hence $\alpha(S) \geq \alpha(T) - n$. But $\alpha(S) < \alpha(T)$. Hence $n \geq 1$.

We deduce that there is an infinite direct oriented simple path in G_S whose origin is an element of T . If we apply this result to the graph obtained from G by reversing the orientation of every edge we find that there is an infinite reverse oriented simple path in G_S whose

origin is an element of T . Hence T is not strictly independent in G_S , contrary to hypothesis.

We conclude that G can have no 1-factor.

4. Hyperprime oriented graphs.

An oriented graph G is *hyperprime* if it is non-null and for each ordered pair $\{a_i, a_j\}$ of distinct vertices, a_i is joined to a_j in G if and only if there is no factorial join from a_i to a_j in G .

A hyperprime graph H is necessarily prime. For if a factor F of H included an edge joining a_i to a_j , there would be both a join and a factorial join from a_i to a_j in H .

(4.1) *If G is any prime finite oriented graph we can construct a hyperprime finite oriented graph H which contains G as a spanning subgraph.*

PROOF. Suppose a_i and a_j are vertices of G ($i \neq j$) such that there is no join and no factorial join from a_i to a_j in G . Then we construct an oriented graph G_1 by adjoining to G a new edge A with positive end a_i and negative end a_j . The graph G_1 is prime. For suppose it has a factor F . Then if A is not an edge of F the graph G has F as a factor, contrary to hypothesis, and if A is an edge of F we obtain a factorial join from a_i to a_j in G from F by deleting the edge A .

If a_p and a_q are vertices of G_1 such that there is no join and no factorial join from a_p to a_q in G_1 , we repeat the process, and so on. Since G is finite we obtain eventually a prime graph H which has G as a spanning subgraph and which has the property that for any two distinct vertices a_i and a_j there is either a join or a factorial join from a_i to a_j .

If an edge A of H joins a_i to a_j there can be no factorial join from a_i to a_j in H . For by adjoining A to such a factorial join we would obtain a 1-factor of H .

We conclude that, since a prime graph is non-null, H is a hyperprime graph having the required properties.

Let H be any hyperprime finite oriented graph.

We partition the vertices of H into four disjoint sets A , B , C , and D , as follows. A is the set of all vertices of H which are out-singular but not in-singular, B is the set of those vertices which are in-singular but not out-singular, C is the set of those vertices which are both in-singular and out-singular, and D is the set of all other vertices of H .

We summarize these definitions in the following diagram.

	Out-singular	Not out-singular
In-singular	C	B
Not in-singular	A	D

(4.2) *Each element of $A \cup C$ is joined to every other vertex of H and each element of $B \cup C$ is joined from every other vertex of H .*

Since H is hyperprime this follows from the definitions of in-singular and out-singular vertices.

(4.3) *No member of $A \cup D$ is joined from any member of $B \cup D$.*

PROOF. Suppose $a_i \in B \cup D$ and $a_j \in A \cup D$, a_i and a_j being distinct. Assume that a_i is joined to a_j in H . There is no factorial join from a_i to a_j since H is hyperprime. Hence either a_i is out-singular or a_j is in-singular by (3.5), contrary to the definitions of these vertices.

(4.4) *The set D is independent in H_C .*

PROOF. If the origin of a nondegenerate direct oriented path in H_C belongs to D , the second vertex and all subsequent vertices must belong to B by (4.3). Hence if the path is finite its terminus cannot be an element of D . The theorem follows.

(4.5) *There exist disjoint sets S and T of vertices of H such that T is independent in H_S and $\alpha(T) > \alpha(S)$.*

PROOF. If $\alpha(D) > \alpha(C)$ the theorem is true, by (4.4). We assume therefore that $\alpha(D) \leq \alpha(C)$.

Let Q be any subset of C such that $\alpha(Q) = \alpha(D)$. If $D \neq 0$ we enumerate the elements of D as d_1, d_2, \dots, d_k , and the elements of Q as q_1, q_2, \dots, q_k . For each pair $\{d_i, q_i\}$ let X_i be a pair of edges of H , one joining d_i to q_i and the other joining q_i to d_i . Such a pair exists, by (4.2).

Suppose first that $\alpha(C) = \alpha(D) = 0$. Then A and B are not both null; otherwise H would be null, contrary to its definition. If $\alpha(A) = 1$ or $\alpha(B) = 1$ then A or B is independent in H_C by (4.3), and so the theorem is true. If neither A nor B has just one element, then for each non-null set A or B we can construct a nondegenerate direct oriented circular path P_A or P_B respectively whose vertices are the elements of A or B respectively. This follows from (4.2). The edges of the resulting path or paths then define a 1-factor of H , contrary to the definition of H .

Suppose next that $\alpha(C) = \alpha(D) \neq 0$. Then we can find a nondegenerate direct oriented circular path P in H in which the vertices are, in order, q_1 , the vertices of $C - Q$ if any, the vertices of A if any, d_1 , the vertices of B if any, and then q_1 again. Let X be the set of edges of P . Clearly the union of X and those sets X_i for which $i > 1$ is the set of edges of a 1-factor of H , contrary to the definition of H .

In the remaining case we have $\alpha(C) > \alpha(D)$. Let x be any element of $C - Q$. We construct a direct oriented circular path P in H according to the following rule. We form its sequence of vertices by taking first x , then the other vertices of $C - Q$ if any, then the vertex q_1 if $D \neq 0$, then the vertices of A if any, then the vertex d_1 if $D \neq 0$, then

the vertices of B if any, and then x again except that x is not to be repeated if it is the only vertex to occur in the sequence. By (4.2) we can construct a direct oriented circular path P with this sequence of vertices. Let its set of edges be X . If P is nondegenerate, the union of X and those sets X_i for which $i > 1$ is the set of edges of a 1-factor of H , contrary to the definition of H .

We deduce that P is degenerate. This implies $A = B = D = 0$ and $\alpha(C) = 1$. Then C is independent in H_D and so the theorem is true.

5. 1-factors of finite oriented graphs.

(5.1) *A finite oriented graph G has no 1-factor if and only if there exist disjoint sets S and T of vertices of G such that T is independent in G_S and $\alpha(S) < \alpha(T)$.*

PROOF. Suppose there are sets S and T of vertices of G with these two properties. Since G is finite the sets S and T are finite and T is strictly independent in G_S . Hence G has no 1-factor, by (3.6).

Conversely suppose G has no 1-factor. By (4.1) we can construct a hyperprime finite oriented graph H which contains G as a spanning subgraph. By (4.5) there are disjoint sets S and T of vertices of H , that is vertices of G , such that $\alpha(T) > \alpha(S)$ and T is independent in H_S . But G_S is a spanning subgraph of H_S . Hence T must be independent in G_S .

6. 1-factors of locally finite oriented graphs. Let G be any infinite but locally finite oriented graph. Let E be its set of edges and V its set of vertices.

A spanning subgraph H of G is *factor-like* on a subset Z of V if $d'(z) = d''(z) = 1$ in H for each $z \in Z$.

(6.1) *Let Z be any finite subset of V . Then either there exists a spanning subgraph H of G which is factor-like on Z or there exist finite subsets S and T of V , T being a subset of Z , such that T is strictly independent in G_S and $\alpha(S) < \alpha(T)$.*

PROOF. Let Q be the set of all elements of $V - Z$ joined to or from elements of Z by edges of G . Let R be any finite subset of $V - (Q \cup Z)$ such that $\alpha(Q \cup R) \geq \alpha(Z) + 2$.

We construct a graph K from $G[Q \cup R \cup Z]$ by adjoining a new edge joining a to b for each ordered pair $\{a, b\}$ of distinct elements of $Q \cup R$.

If K has a 1-factor F let U be the set of those edges of F which have an element of Z as one end. The spanning subgraph of G whose edges are the elements of U is factor-like on Z .

If K has no 1-factor there are disjoint subsets S and T of vertices of K such that T is independent in K_S and $\alpha(S) < \alpha(T)$, by (5.1).

Suppose T includes an element x of $Q \cup R$. Since T is independent in K_S it follows from the construction for K that $(Q \cup R) - \{x\} \subseteq S$. Hence $\alpha(Q \cup R) \leq \alpha(S) + 1 \leq \alpha(T)$. But we have also $\alpha(Q \cup R) \geq \alpha(Z) + 2 \geq \alpha(T) + 1$. From this contradiction we conclude that $T \subseteq Z$.

Suppose that T is not strictly independent in G_S . Since T is independent in K_S it follows that there exist direct oriented simple paths P_1 and P_2 in G_S , the first from an element of T to an element a of Q and the second from another element b of Q to an element of T , each path having only one term which is in $Q \cup R$. But a is joined to b by an edge of K . It follows that there is a finite nondegenerate direct oriented path in K_S whose origin and terminus are elements of T . This is impossible since T is independent in K_S .

The theorem follows.

(6.2) *A locally finite oriented graph G has no 1-factor if and only if there exist disjoint finite sets S and T of vertices of G such that T is strictly independent in G_S and $\alpha(S) < \alpha(T)$.*

PROOF. Suppose there are sets S and T of vertices of G with these properties. Then G has no 1-factor, by (3.6).

Conversely suppose G has no 1-factor.

If G is finite the theorem holds, by (5.1). We suppose therefore that G is infinite. We begin by considering the case in which G is infinite and connected. The sets E and V of edges and vertices respectively of G are then denumerable, by (2.2).

Let the vertices of G be enumerated as (a_1, a_2, a_3, \dots) . For each positive integer n we denote the set $\{a_1, a_2, \dots, a_n\}$ by Z_n . Assume that for each n there is a spanning subgraph H_n of G which is factor-like on Z_n .

Let the edges of G be enumerated as (A_1, A_2, A_3, \dots) . Write $f(m, n) = 1$ if A_m is an edge of H_n , and $f(m, n) = 0$ otherwise. If Σ_1 denotes the infinite sequence (H_1, H_2, H_3, \dots) there must be an infinite subsequence Σ_2 of Σ_1 such that $f(1, n)$ has the same value, $f(1)$ say, for each n such that $H_n \in \Sigma_2$. Further there must be an infinite subsequence Σ_3 of Σ_2 such that $f(2, n)$ has the same value, $f(2)$ say, for each n such that $H_n \in \Sigma_3$, and so on. Accordingly there is an infinite strictly increasing sequence (n_1, n_2, n_3, \dots) of positive integers such that

$$\lim_{i \rightarrow \infty} f(m, n_i) = f(m)$$

for each positive integer m . (We may define n_j as the first suffix in Σ_j exceeding n_{j-1} .) Let F be the spanning subgraph of G whose edges are those edges A_m of G for which $f(m) = 1$. Then F is factor-like on

each of the sets Z_n and is therefore a 1-factor of G , contrary to assumption.

We conclude that there is a positive integer q such that no spanning subgraph of G is factor-like on Z_q . An application of (6.1) completes the proof for the case in which G is connected.

In the remaining case G is not connected. If each component of G has a 1-factor, then G has a 1-factor, the union of one 1-factor from each component. We deduce that some component H of G has no 1-factor. By the preceding result there are disjoint finite sets S and T of vertices of H such that T is strictly independent in H_S , and $\alpha(S) < \alpha(T)$. But if T is strictly independent in H_S it is necessarily strictly independent in G_S . This completes the proof of the theorem.

7. An application to unoriented graphs. The *degree* of a vertex a in an unoriented graph G is the number of edges of G having a as an end. G is *locally finite* if each vertex of G has a finite degree. A locally finite graph G is *regular of degree n* if all its vertices have the same degree n .

Let us define a *Q-factor* of an unoriented graph G as a subgraph of G including all the vertices of G in which each component is regular of degree 1 or regular of degree 2. A component of the *Q-factor* which is regular of degree 1 must have just one edge and just two vertices.

(7.1) *A locally finite unoriented graph G has no Q-factor if and only if there are disjoint finite sets S and T of vertices of G such that $\alpha(S) < \alpha(T)$ and each edge of G which has one end in T has the other end in S .*

PROOF. If G has an edge A joining two vertices a and b we replace it by two oriented edges, one joining a to b and the other joining b to a . Making this substitution for each edge we convert G into a locally finite oriented graph H . Clearly G has a *Q-factor* if and only if H has a 1-factor. Hence it follows from (6.2) and the construction for H that (7.1) is true.

An unoriented graph G is *even* if its set of vertices can be partitioned into two disjoint subsets V_1 and V_2 such that each edge has one end in V_1 and the other in V_2 . If an even graph G has a *Q-factor* Q then it has a *Q-factor* Q' which is regular of degree 1. We may derive a Q' from Q by suppressing alternate edges in each component of Q which is regular of degree 2. The *Q-factor* Q' is a 1-factor of G in the sense of the theory of unoriented graphs. Accordingly for an even graph (7.1) is equivalent to the theorem of P. Hall and R. Rado on the factorization of even graphs [2; 3].

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ON SEMI-SIMPLE LIE ALGEBRAS

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1. The primary purpose of this note is to give a new proof for the sufficiency of E. Cartan's criterion for semi-simplicity of a Lie algebra, namely that its Killing form be nondegenerate. My proof differs from the usual ones in the fact that it uses no result from the theory of nilpotent Lie algebras, and is valid for a base field of *arbitrary characteristic*.

2. Let \mathfrak{g} be a Lie algebra over a field K , having finite dimension $n > 0$. A symmetric bilinear form $\phi(X, Y)$ over $\mathfrak{g} \times \mathfrak{g}$ is called *invariant* if $\phi([X, Y], Z) = \phi(X, [Y, Z])$ identically. This is the case for the *Killing form* $\text{Tr}(\text{ad}(X) \text{ad}(Y))$, where $\text{ad}(X)$ is the endomorphism $Y \rightarrow [X, Y]$ of the vector space \mathfrak{g} . It is well known that when the Killing form of \mathfrak{g} is nondegenerate, \mathfrak{g} does not contain any abelian ideal $\neq (0)$ (one has only to remark that if \mathfrak{a} is such an ideal, and $A \in \mathfrak{a}$, then $(\text{ad}(A) \text{ad}(X))^2 = 0$ for any $X \in \mathfrak{g}$, by an elementary computation, hence $\text{Tr}(\text{ad}(A) \text{ad}(X)) = 0$ for all $X \in \mathfrak{g}$). E. Cartan's criterion is therefore a consequence of the more general result:¹

THEOREM. *If the Lie algebra \mathfrak{g} does not contain any abelian ideal $\neq (0)$, and if there exists a symmetric invariant nondegenerate bilinear form $\phi(X, Y)$ on $\mathfrak{g} \times \mathfrak{g}$, then \mathfrak{g} is a direct sum of simple nonabelian subalgebras.*

Let \mathfrak{m} be a minimal ideal in \mathfrak{g} ; as $[\mathfrak{m}, \mathfrak{m}]$ is an ideal of \mathfrak{g} , contained in \mathfrak{m} , $[\mathfrak{m}, \mathfrak{m}]$ is either (0) or \mathfrak{m} ; but the first case is excluded, since

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¹ I am indebted to N. Jacobson for calling my attention to this generalization, as well as for simplifying my original proof.