ON THE BEHAVIOR OF FOURIER COEFFICIENTS

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1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π , and let

$$(1) \quad f(t) \sim \frac{1}{2} a_0 + \sum_1^{\infty} (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_1^{\infty} A_n(t).$$

Then the conjugate series of (1) at $t=x$ is

$$(2) \quad \sum_1^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_1^{\infty} B_n(x).$$

We write

$$(3) \quad \theta(t) = f(x+t) - f(x-t), \quad \psi(t) = f(x+t) - f(x-t) - l,$$

where l is a finite number.

It is known¹ that if $\psi(t) \rightarrow 0$ as $t \rightarrow 0$, then the sequence $\{nB_n(x)\}$ is summable (C, r) , $r > 1$, to the value l/π , but not necessarily summable $(C, 1)$. It is also known² that with the same hypothesis the sequence $\{nB_n(x)\}$ is summable by the first logarithmic mean to the value l/π .

The object of this note is to prove the following

THEOREM 1. *If*

$$(4) \quad \psi(t) = o\left\{\left(\log \frac{1}{t}\right)^{-1}\right\} \quad \text{as } t \rightarrow 0$$

and

$$(5) \quad a_n = O(n^{-\delta}), \quad b_n = O(n^{-\delta}), \quad 0 < \delta < 1,$$

then the sequence $\{nB_n(x)\}$ is summable $(C, 1)$ to the value l/π .

2. Proof of the theorem. We are to prove that

$$\frac{1}{n} \sum_1^n rB_r(x) \rightarrow \frac{l}{\pi} \quad \text{as } n \rightarrow \infty.$$

Writing $t_n(x) = \sum_1^n rB_r(x)$, we have

Received by the editors November 19, 1952 and, in revised form, June 30, 1953.

¹ See Zygmund, *Trigonometrical series*, 1935, p. 62.

² Zygmund, loc. cit.

$$\frac{t_n(x)}{n} = \frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t)\} g(n, t) dt,$$

where

$$g(n, t) = \frac{1}{n} \sum_1^n r \sin rt.$$

Denoting $\int_0^\pi g(n, t) dt$ by λ_n , we have

$$\lambda_n = \frac{1}{n} \sum_1^n \{1 - (-1)^r\} = 1 + o(1), \quad \text{as } n \rightarrow \infty.$$

Therefore

$$\begin{aligned} \frac{1}{n} t_n(x) - \frac{l}{\pi} \lambda_n &= \frac{1}{\pi} \int_0^\pi \{f(x+t) - f(x-t) - l\} g(n, t) dt \\ &= \frac{1}{\pi} \int_0^\pi \psi(t) g(n, t) dt. \end{aligned}$$

Thus

$$\begin{aligned} (6) \quad \frac{1}{n} t_n(x) - \frac{l}{\pi} &= \frac{1}{\pi} \int_0^\pi \psi(t) g(n, t) dt + o(1) \\ &= J + o(1), \quad \text{say.} \end{aligned}$$

We have

$$\begin{aligned} g(n, t) &= -\frac{1}{n} \frac{d}{dt} \{1/2 + \cos t + \cos 2t + \cdots + \cos nt\} \\ &= -\frac{1}{n} \frac{d}{dt} \left\{ \frac{\sin(n+1/2)t}{2 \sin(t/2)} \right\} \\ &= \frac{1}{4n} \frac{\sin(n+1/2)t \cos(t/2)}{\sin^2(t/2)} - \frac{(n+1/2)}{2n} \frac{\cos(n+1/2)t}{\sin(t/2)}. \end{aligned}$$

Let

$$(7) \quad J = \int_0^\pi \psi(t) g(n, t) dt = \int_0^{n^-} + \int_{n-1}^{n^-} + \int_{n-r}^\pi = J_1 + J_2 + J_3,$$

where $r = \delta/2$. We require the following inequalities for $g(n, t)$.

$$(8) \quad g(n, t) = O(n^2 t),$$

$$(9) \quad g(n, t) = O(t^{-1}).$$

PROOF OF (8).

$$g(n, t) = \frac{\sin(n + 1/2)t \cos(t/2) - 2(n + 1/2) \cos(n + 1/2)t \sin(t/2)}{4n \sin^2(t/2)}.$$

By replacing the sines and cosines in the numerator of the above fraction by their respective power series expansions it is easy to see that, for $0 < t < 1/n$, the numerator is $O(n^3 t^3)$ and hence

$$|g(n, t)| < K (n^3 t^3 / nt^2) = Kn^2 t,$$

which completes the proof of (8).

PROOF OF (9). Now

$$\left| \frac{\sin(n + 1/2)t \cos(t/2)}{4n \sin^2(t/2)} \right| < \frac{(n + 1/2)t \pi^2}{4nt^2} < \frac{\pi^2}{2} t^{-1}.$$

Similarly

$$\left| \frac{-(n + 1/2) \cos(n + 1/2)t}{2n \sin(t/2)} \right| < \pi t^{-1}.$$

Therefore,

$$|g(n, t)| < (\pi + \pi^2/2)t^{-1},$$

which completes the proof of (9).

Using (8) we find

$$(10) \quad J_1 = \int_0^{n^{-1}} \psi(t) g(n, t) dt = \int_0^{n^{-1}} o(1) O(n^2 t) dt = o(1).$$

By (9) we have

$$(11) \quad |J_2| = o \left\{ \int_{n^{-1}}^{n^{-r}} \frac{dt}{t \log(1/t)} \right\} = o(1).$$

Now we write

$$(12) \quad J_3 = J_{3.1} - J_{3.2}$$

where

$$(13) \quad J_{3.1} = \frac{1}{4n} \int_{n^{-r}}^{\pi} \psi(t) \frac{\cos(t/2) \sin(n + 1/2)t}{\sin^2(t/2)} dt,$$

$$(14) \quad J_{3.2} = \frac{n + 1/2}{2n} \int_{n^{-r}}^{\pi} \psi(t) \frac{\cos(n + 1/2)t}{\sin(t/2)} dt.$$

We take up $J_{3.2}$ first. Now the Fourier series for $\psi(t)$ is

$$2 \sum_1^{\infty} B_n(x) \sin nt - l.$$

By multiplying the Fourier series of $\psi(t)$ by $\cos (n+1/2)t/\sin (t/2)$ which is of bounded variation and integrating term by term, we have³

$$\begin{aligned} \int_{n-r}^{\pi} \psi(t) \frac{\cos (n+1/2)t}{\sin (t/2)} dt &= \sum_{k=1}^{\infty} B_k \int_{n-r}^{\pi} \frac{2 \sin kt \cos (n+1/2)t}{\sin (t/2)} dt \\ &\quad - l \int_{n-r}^{\pi} \frac{\cos (n+1/2)t}{\sin (t/2)} dt. \end{aligned}$$

Now the coefficient of B_k equals

$$\begin{aligned} &\int_{n-r}^{\pi} \frac{\sin (n+1/2+k)t - \sin (n+1/2-k)t}{\sin (t/2)} dt \\ &= (\operatorname{cosec} (n-r/2)) \int_{n-r}^{\xi} \{\sin (n+1/2+k)t - \sin (n+1/2-k)t\} dt \\ &\quad \quad \quad (n-r < \xi < \pi) \end{aligned}$$

and it follows that the coefficient of B_k does not exceed $4\pi n^r/|k-(n+1/2)|$ in absolute value.

By the second mean value theorem, we find

$$\int_{n-r}^{\pi} \frac{\cos (n+1/2)t}{\sin (t/2)} dt = o(1).$$

Thus, by (14),

$$\begin{aligned} |J_{3.2}| &< o(1) + 4\pi \sum_{k=1}^{\infty} \frac{k^{-\delta} n^r}{|k-(n+1/2)|} = o(1) + \sum_{k=1}^n + \sum_{k=n+1}^{\infty} \\ &\equiv o(1) + R_1 + R_2. \end{aligned}$$

Now

$$\begin{aligned} \frac{R_1}{4\pi} &< \frac{n^r}{(n+1)/2} \sum_{1 \leq k \leq n/2} k^{-\delta} + n^r \left(\frac{n}{2}\right)^{-\delta} \sum_{n/2 \leq k \leq n} \frac{1}{(n+1/2)-k} \\ &= O(n^{-\delta/2}) + O(n^{-\delta/2} \log n) = o(1), \end{aligned}$$

and

$$\begin{aligned} \frac{R_2}{4\pi} &< n^r n^{-\delta} \sum_{n < k \leq 2n} \frac{1}{k-(n+1/2)} + 2n^r \sum_{k > 2n} \frac{k^{-\delta}}{k-1} \\ &= O(n^{-\delta/2} \log n) + O(n^{-\delta/2}) = o(1). \end{aligned}$$

³ Zygmund, *Trigonometrical series*, 1935. The discussion of $J_{3.2}$ is effected by arguments similar to those in Zygmund and has been given only for the sake of completeness.

Hence

$$(15) \quad J_{3,2} = o(1).$$

Again

$$J_{3,1} = \frac{1}{2n} \int_{n-\tau}^{\pi} \frac{\psi(t) \{ \sin(n+1)t + \sin nt \} dt}{\sin^2(t/2)} = I_3 + I_4,$$

where

$$(16) \quad I_3 = \frac{1}{2n} \int_{n-\tau}^{\pi} \psi(t) \frac{\sin(n+1)t}{\sin^2(t/2)} dt$$

and

$$(17) \quad I_4 = \frac{1}{2n} \int_{n-\tau}^{\pi} \psi(t) \frac{\sin nt}{\sin^2(t/2)} dt.$$

By applying arguments similar to that used above it can be easily shown that

$$I_3 = O\left(\frac{\log n}{n}\right) = o(1)$$

and

$$I_4 = O\left(\frac{\log n}{n}\right) = o(1).$$

Hence

$$(18) \quad J_{3,1} = o(1).$$

Therefore by (12), (18), and (15) we have

$$(19) \quad J_3 = o(1).$$

By (7), (10), (11), and (19), we have

$$(20) \quad J = o(1).$$

Therefore by (6) and (20),

$$\frac{1}{n} t_n(x) - \frac{l}{\pi} = o(1) \quad \text{as } n \rightarrow \infty,$$

and this completes the proof of the theorem.

3. It is easy to see from Schwarz's example of a divergent Fourier series at a point of continuity slightly modified that condition (4)

alone does not imply summability $(C, 1)$ of the sequence $nB_n(x)$.

Hardy-Littlewood's convergence criterion for the allied series is⁴

THEOREM A. *If*

$$A(i) \quad \theta(t) = o \left\{ \left(\log \frac{1}{t} \right)^{-1} \right\}$$

and

$$A(ii) \quad \text{each of } a_n \text{ and } b_n \text{ is } O(n^{-\delta}), \quad 0 < \delta < 1,$$

then the allied series $\sum_1^\infty B_n(x)$ converges to the value

$$(21) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \theta(t) \cot(t/2) dt,$$

provided that the integral exists as a "Cauchy integral" at the origin.

We shall now deduce Theorem A as a corollary of our theorem by employing the following known result also due to Hardy and Littlewood.⁵

THEOREM B. *If $\sum u_n$ is summable (A), then a necessary and sufficient condition that it should be summable (C, k) , $k > -1$, is that the sequence $\{nu_n\}$ is summable $(C, k+1)$ to the value 0.*

PROOF OF THEOREM A. The existence of the integral (21) as a "Cauchy integral" at the origin implies the summability (A) of the allied series $\sum_1^\infty B_n(x)$.⁶

By using Theorem 1, we find that conditions A(i) and A(ii) of Theorem A imply the summability $(C, 1)$ of the sequence $\{nB_n(x)\}$ to the value 0. Now the convergence of the allied series is a consequence of Theorem B.

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⁴ Hardy and Littlewood, *Annali della R. Scuola Normale Superiori di Pisa* (2) vol. 3 (1934) pp. 43-62, Theorem 13. See also Zygmund, *Trigonometrical series*, 1935, p. 35, where the slightly stronger theorem, viz.: If (i) $f(x+h) - f(x) = O(\log 1/|h|)^{-1}$, and (ii) each of a_n and b_n is $O(n^{-\delta})$, $\delta > 0$, then $\bar{S}_n(x) - \bar{f}(x, \pi/n) = o(1)$ (in the notation of Zygmund), is given, which includes Theorem A.

⁵ Hardy and Littlewood, *J. London Math. Soc.* vol. 6 (1931) p. 283.

⁶ Zygmund, *Trigonometrical series*, 1935, p. 55.