2. M. Ikeda and T. Nakayama, *Supplementary remarks on Frobeniusean algebras*, II, Osaka Math. J. vol. 2 (1950) pp. 7–12.

3. T. Nakayama, *On Frobeniusean algebras*, II, Ann. of Math. vol. 42 (1941) pp. 1–21.

OSAKA UNIVERSITY AND
NAGOYA UNIVERSITY

THE NUMBER OF SUBGROUPS OF GIVEN INDEX IN NONDENUMERABLE ABELIAN GROUPS

W. R. SCOTT

Let G be an Abelian group of order $A > \aleph_0$. It has been shown [4, Theorem 9] that there exist 2^A subgroups of G of order A, and that the intersection of all such subgroups is 0. In this paper, this result is improved to the following: If $\aleph_0 \leqq B \leqq A$ and $A > \aleph_0$, then an Abelian group of order A has 2^A subgroups of index B, and the intersection of all such subgroups is 0. In addition, it is shown that there is a set of 2^A subgroups H_α of index B such that $G/H_\alpha \cong G/H_{\alpha'}$ for all α, α'.

Baer [1, p. 124] showed that if G is an Abelian p-group which is the direct sum of A cyclic groups of bounded order, then G has 2^A subgroups of index p (here A may equal $\aleph_0$). The proof in the present paper is accomplished by extending Baer's result in an obvious manner to a wider class of p-groups, and then reducing all other cases to this one.

We shall use $+$ and $\sum$ to denote direct sums, and $o(S)$ to denote the number of elements in S.

LEMMA. *Let $H \neq 0$ be an Abelian group, and let $G = \sum H_\alpha$, $\alpha \in S$, $H_\alpha \cong H$ for all $\alpha \in S$, $o(S) = A \geqq \aleph_0$. Then there are at least 2^A subgroups K_β of G such that $G/K_\beta \cong H$.*

PROOF. (This proof is the same as Baer's, and is included only for the sake of completeness.) Identify H_α with H. Let ϵ_α be 0 or 1 for each α, $\alpha \in S$. Let K be the set of elements of G such that $h_{\alpha_0} = \sum \epsilon_\alpha h_\alpha$, $\alpha \neq \alpha_0$. Then it is easy to verify that K is a subgroup of G and $G = H_{\alpha_0} + K$. If $\epsilon_\alpha = 0$, then $H_\alpha \subset K$, but if $\epsilon_\alpha = 1$, then $H_\alpha \cap K = 0$. Thus all of the K's are distinct, and the lemma is proved.

Presented to the Society, September 3, 1953; received by the editors March 2, 1953.

THEOREM. *Let* G *be an Abelian group of order* $A>\aleph_0$, *and let* $\aleph_0\leqq B\leqq A$. *Then*

(i) *there are exactly* 2^A *subgroups* H_α *of index* B *and order* A,

(ii) *the intersection of the subgroups in* (i) *is* 0,

(iii) *there exist* 2^A *subgroups* K_β, *of index* B *and order* A, *such that* $G/K_\beta\cong G/K_{\beta'}\cong\sum C_\gamma$ *where either*

(a) *all* C_γ *are cyclic of prime order, or*

(b) *all* C_γ *are* p^∞ *groups* (p *not necessarily fixed*).

REMARKS. The condition that $o(H_\alpha)=A$ or $o(K_\beta)=A$ is automatically satisfied by subgroups of index B unless $B=A$. If (ii) is true, then it is clear that there is a set of A subgroups, each of index B and order A, whose intersection is 0. Finally, since there are at most 2^A subgroups of G, (iii) implies (i), hence only (ii) and (iii) need be proved.

PROOF. Case 1. $G=\sum C_\gamma$, $\gamma\in S$, $o(C_\gamma)=p$, p a fixed prime. Then $o(S)=A$. By omitting B and retaining A summands, one obtains a subgroup K of index B and order A. By the lemma, K has 2^A subgroups K_β of index p in K. The K_β are thus of order A and of index B in G, and (iii) is satisfied. Since any given summand could have been omitted in obtaining K, (ii) is satisfied.

Case 2. $G=\sum C_\gamma$, $\gamma\in S$, where C_γ is cyclic of order p^{n_γ}. Again $o(S)=A$, and G/pG is of the type considered in Case 1. Hence there are 2^A subgroups K^*_β of G/pG as in (iii). Therefore there are 2^A subgroups K_β of G satisfying (iii). To obtain (ii), first omit from G one summand containing a nonzero component of a given nonzero element g. Since the group G^* thus obtained has finite index in G, one may then proceed as above.

Case 3. $G=\sum C_\gamma$, $\gamma\in S$, where C_γ is a p^∞ group, p fixed. The proof is nearly identical to that in Case 1.

Case 4. G is a p-group. Then [**3**, Theorem 6] there exists a pure (=servant) subgroup M of G such that (α) $M=\sum C_\gamma$, $\gamma\in S$, where C_γ is cyclic, and (β) $G/M=\sum D_\delta$, where the D_δ are p^∞ groups. If $\gamma_1,\cdots,\gamma_n\in S$, then $C_{\gamma_1}+\cdots+C_{\gamma_n}$ is a pure subgroup of M since it is a direct summand thereof, consequently $C_{\gamma_1}+\cdots+C_{\gamma_n}$ is a pure subgroup of G. But since it is also of bounded order, we have [**2**, Theorem 5] (γ) $C_{\gamma_1}+\cdots+C_{\gamma_n}$ is a direct summand of G for all $\gamma_1,\cdots,\gamma_n\in S$.

If $o(G/M)=A$, then by (β) and Case 3, (iii) is satisfied for G/M, hence also for G.

If $o(G/M)<A$, then $o(M)=A$, and if V is a set of representatives of the cosets of M, then the subgroup L generated by V has order less than A, and $M\cup L=G$. The summands C_γ containing any com-

ponent of any element of $M\cap L$ are fewer than A in number. Hence there exists a subgroup N of M such that $N\supset M\cap L$, $o(M/N)=A$, and M/N is isomorphic to a direct sum of some of the C_γ. Therefore by Case 2, (iii) is true for M/N, and therefore there are 2^A subgroups K_β^* of M containing N, and therefore $M\cap L$, such that the factor groups M/K_β^* are as in (iii). It then follows from the isomorphism theorem that (iii) is satisfied for G.

To prove (ii), note that by (γ), any element of M may be omitted by a subgroup of finite index in G, hence from the above, by a subgroup H_α of index B and order A. If $o(G/M)>B$, then by (β) and Case 3 (perhaps for a cardinal smaller than A), any nonzero element of G/M may be omitted by a subgroup of order $o(G/M)$ and of index B in G/M, hence any element of G outside of M may be omitted by a subgroup of order $o(G/M)o(M)=A$ and of index B in G. The same is true if $o(G/M)=B=A$. If $o(G/M)<B$ or $o(G/M)=B<A$, then any subgroup of M of order A and index B in M (omit B and keep A summands C_γ in (α)) has the property of omitting all elements outside M and of having the right order and index in G.

Case 5. G is periodic. Then $G=\sum G_p$, where G_p is the p-component of G, and $A=\sum o(G_p)$. Let

$$\begin{aligned} S_1 &= \{p \mid o(G_p) \leqq \aleph_0\}, \\ S_2 &= \{p \mid o(G_p) > \aleph_0, \text{ and (iii)(a) holds for } G_p\}, \\ S_3 &= \{p \mid o(G_p) > \aleph_0, \text{ and (iii)(a) does not hold for } G_p\}, \\ N_i &= \sum_{p\in S_i} G_p, \qquad i = 1, 2, 3. \end{aligned}$$

Then either $o(N_2)=A$ or $o(N_3)=A$. Now if $p\in S_2$, then by Case 4, G_p has $2^{o(G_p)}$ subgroups $K_{p,\gamma}$ of order $o(G_p)$ and index min $(o(G_p), B)$ such that the $G_p/K_{p,\gamma}$ are as in (iii)(a). Hence if $K=N_1+N_3+\sum K_{p,\gamma}$, $p\in S_2$, then G/K is isomorphic to a fixed product of type (a) for all choices of K, and K is of order A and of index min $(o(N_2), B)$ in G. There are

$$\prod_{p\in S_2} 2^{o(G_p)} = 2^{\Sigma_{p\in S_1} o(G_p)} = 2^{o(N_2)}$$

such K. Similarly there are (at least) $2^{o(N_3)}$ K_β such that K_β is of order A, of index min $(o(N_3), B)$ in G, and such that G/K_β is of type (b). Hence (iii) is satisfied, since either $o(N_2)$ or $o(N_3)$ equals A.

To prove (ii), let $g\in G$ have component $g_{p_0}\neq 0$. If $o(G_{p_0})<B$, or $o(G_{p_0})=B<A$, then omit the summand G_{p_0} from G, and find a subgroup of order A and of index B in the resulting G^*. If $o(G_{p_0})>B$ or $o(G_{p_0})=B=A$, then by Case 4, there exists a subgroup K_{p_0} of G_{p_0}

of index B in G_{p_0} and, in case $B=A$, of order A, which omits g_{p_0}. Then $K_{p_0}+\sum G_p$, $p\neq p_0$, omits g and has the required properties. Thus (ii) holds.

Case 6. G is not periodic. Let $F=\{f_\alpha\}$ be a maximal independent set of elements of G. Let L_k, $k=2, 3, \cdots$, be the subgroup generated by the maximal independent set of elements $\{kf_\alpha\}$. If $o(F)<A$, then $o(L_k)<A$ and $o(G/L_k)=A$. If $o(F)=A$, then the f_α lie in distinct cosets of L_k, so again $o(G/L_k)=A$. But G/L_k is periodic, and therefore by Case 5, (iii) is true, and in (ii), $\bigcap H_\alpha\subseteq L_k$. Since $\bigcap L_k=0$, $k=2, 3, \cdots$, (ii) also holds.

BIBLIOGRAPHY

1. R. Baer, *Dualism in abelian groups*, Bull. Amer. Math. Soc. vol. 43 (1937) pp. 121–124.

2. L. Koulikoff, *Zur Theorie der abelschen Gruppen von beliebiger Mächtigkeit*, Mat. Sbornik vol. 9 (1941) pp. 165–181 (Russian with Germany summary).

3. ——— *On the theory of abelian groups of arbitrary power*, Mat. Sbornik vol. 16 (1945) pp. 129–162 (Russian with English summary).

4. W. R. Scott, *Groups and cardinal numbers*, Amer. J. Math. vol. 74 (1952) pp. 187–197.

UNIVERSITY OF KANSAS