NOTE ON THE HEAT EQUATION¹

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If $u(x, t)$ is the temperature of an infinite insulated rod, then

$$(1) \quad H(a, b; t) = \int_a^b u(x, t) dx$$

is the amount of heat at time t in the interval $[a, b]$. We prove an existence and uniqueness theorem for the case in which the heat, rather than the temperature, is prescribed initially. If the prescribed heat is not absolutely continuous, its derivative $\partial H/\partial b$, the initial temperature, may not determine H , so that we are dealing with a more general problem than the usual one.

The following lemma, which we did not find in the literature, will be useful.

LEMMA 1. *Let $f(x)$ be measurable, and $|f(x)| \leq Me^{cx^2}$. Define*

$$(2) \quad W_r(x, t) = (4\pi t)^{-1/2} \frac{\partial^r}{\partial x^r} [e^{-x^2/4t}].$$

Then, for every integer $r \geq 0$, the integral

$$(3) \quad u_r(x, t) = \int_{-\infty}^{\infty} W_r(x - \xi, t) f(\xi) d\xi$$

is absolutely convergent and satisfies

$$(a) \quad \partial u_r / \partial x = u_{r+1}, \quad \partial u_r / \partial t = u_{r+2} \quad \text{on } 0 < t < 1/4c,$$

and, for all t_0 satisfying $0 < t_0 < 1/4c$,

$$(b) \quad |u_r(x, t)| \leq K t^{-r/2} e^{N x^2}, \quad \text{on } 0 < t \leq t_0,$$

where $K = K(r, t_0)$, and $N = N(t_0)$.

PROOF. It is well known that $W_r(x, t)$ is analytic, satisfies the heat equation, and has the specific form

$$(4) \quad W_r(x, t) = \frac{1}{(2\pi)^{1/2}} (2t)^{-(r+1)/2} H_r\left(\frac{x}{(2t)^{1/2}}\right) e^{-x^2/4t},$$

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where $H_r(z)$ is the Hermite polynomial of degree r in $z=x/(2t)^{1/2}$. The absolute convergence of (3) on $0 < t < 1/4c$ is obvious from (4) and $|f(x)| \leq Me^{cx^2}$.

To prove the first equation of (a), note

$$(5) \quad h^{-1}[u_r(x+h, t) - u_r(x, t)] = \int_{-\infty}^{\infty} W_{r+1}(x+\theta h - \xi, t)f(\xi)d\xi,$$

$$0 \leq \theta \leq 1,$$

by the Law of the Mean. But, as $|\xi| \rightarrow \infty$,

$$|W_{r+1}(x+\theta h - \xi, t)| = O(|x - \xi| + 1)^{r+1}e^{-(|x-\xi|+1)^2/4t},$$

and is bounded in finite intervals. Hence, since $t < 1/4c$, the integrand in (5) tends *dominatedly* [3, p. 168] to $W_{r+1}(x-\xi, t)f(\xi)$ as $h \rightarrow 0$, which shows that

$$\lim_{h \rightarrow 0} h^{-1}[u_r(x+h, t) - u_r(x, t)] = \int_{-\infty}^{\infty} W_{r+1}(x - \xi, t)f(\xi)d\xi,$$

proving the first equation of (a). A similar proof applies to the second equation.

To prove (b), note that the right side of (3) is by (4) bounded by a sum of terms of the form

$$Mt^{-s} \int_{-\infty}^{\infty} |x - \xi|^s e^{-(x-\xi)^2/4t + c\xi^2} d\xi \quad [s \leq r].$$

Completing the square, the exponent becomes

$$[(1-4ct)/4t][\xi - x/(1-4ct)]^2 + cx^2/(1-4ct).$$

Writing $\eta = t^{-1/2}[\xi - x/(1-4ct)]$, and noting that $c/(1-4ct) < c/(1-4ct_0) = N'(t_0)$, we bound the right side of (3) by a sum of terms

$$M'e^{N'x^2}t^{-s} \int_{-\infty}^{\infty} |\eta t + N'xt|^s e^{-(1-4ct)\eta^2} d\eta.$$

Since $1-4ct \geq 1-4ct_0 = L > 0$, another bound is

$$M'e^{N'x^2}t^{-s/2} \int_{-\infty}^{\infty} |\eta + N'xt^{1/2}|^s e^{-L\eta} d\eta.$$

For any $N > N'(t_0)$ and suitable $K = K(t_0, r)$, this evidently implies (b).

COROLLARY. In Lemma 1, $u_r \in C^\infty$ and $\partial^{m+n}u_r/\partial x^m \partial t^n = u_{r+m+2n}$.

We shall also use the following essentially known facts.

THEOREM A. *Let $f(x)$ be measurable, and let $|f(x)| \leq Me^{cx^2}$, $c > 0$. Then*

$$(6) \quad u(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t) f(\xi) d\xi,$$

is absolutely convergent for $0 < t < 1/4c$, and

$$(\alpha) \quad u(x, t) \in C^\infty \quad \text{and} \quad u_{xx} = u_t, \quad 0 < t < 1/4c,$$

$$(\beta) \quad |u(x, t)| \leq Ke^{Nx^2}, \quad 0 < t \leq t_0 < 1/4c, \quad K = K(t_0), \quad N = N(t_0).$$

If $f(x)$ has bounded variation on every finite interval, and $2f(x) = f(x^+) + f(x^-)$, then

$$(\gamma) \quad \lim_{t \rightarrow 0^+} u(x, t) = f(x).$$

If $f(x)$ is continuous, then

$$(\gamma') \quad \lim_{t \rightarrow 0^+, x \rightarrow x_0} u(x, t) = f(x_0),$$

so that if $u(x, 0)$ is defined as $f(x)$, then $u(x, t)$ is continuous for $0 \leq t < 1/4c$.

PROOF. For (α) , (γ) , and (γ') see [1, p. 298]; (β) is part of Lemma 1.

THEOREM B. *Conversely, if $u(x, t)$ is defined for $0 < t \leq t_0$ and satisfies*

$$(\alpha) \quad u(x, t) \in C^2 \quad \text{and} \quad u_{xx} = u_t, \quad 0 < t \leq t_0,$$

$$(\beta) \quad |u(x, t)| \leq Ke^{Nx^2}, \quad 0 < t \leq t_0, \quad \text{for some finite constants}$$

$$K = K(t_0), \quad N = N(t_0),$$

$$(\gamma) \quad \lim_{t \rightarrow 0^+} |u(x, t)| = 0, \quad \text{for all } x,$$

then

$$u(x, t) \equiv 0, \quad 0 < t \leq t_0.$$

The usual theorem, assuming simultaneous continuity in x and t , $t \geq 0$, is in [2, p. 88]. To prove our slightly sharper result, note that by the usual theorem, if $0 < \tau < t < 1/4N$, then

$$u(x, t) = \lim_{\tau \rightarrow 0^+} \int_{-\infty}^{\infty} u(\xi, \tau) W_0(x - \xi, t - \tau) d\xi.$$

If $\tau < t/2$, then $W_0(x - \xi, t - \tau) \leq (2\pi t)^{-1/2} e^{-(x - \xi)^2/2t}$. Hence (β) gives dominated convergence [3, p. 168], and we can pass to the limit

under the integral sign. We conclude that $u(x, t) = 0$ for all t such that $t < 1/4N$. Suppose t_1 is the l.u.b. of the values of t for which $\tau < t \rightarrow u(x, \tau) = 0$, and suppose $t_1 < t_0$. Then, by continuity, $u(x, t_1) = 0$, and we may repeat the preceding argument to prove that $u(x, t) = 0$, $t \leq t_1 + \eta$, for some positive η . This contradiction shows that $t_1 = t_0$, completing the proof of Theorem 2.

We shall now prove our existence theorem.

THEOREM 1. *Let $H(x)$ have bounded variation on every finite interval, let $2H(x) = H(x^+) + H(x^-)$, $H(0) = 0$, and let $|H(x)| \leq Me^{cx^2}$. Then the improper integral*

$$(7) \quad u(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t) dH(\xi) = \lim_{A \rightarrow \infty} \int_{-A}^A W_0(x - \xi, t) dH(\xi)$$

exists for $0 < t < 1/4c$, and satisfies

$$(a) \quad u(x, t) \in C^\infty \quad \text{and} \quad u_{xx} = u_t \quad \text{on} \quad 0 < t < 1/4c,$$

$$(b) \quad |u(x, t)| \leq Kt^{-1/2}e^{Nx^2} \quad \text{on} \quad 0 < t \leq t_0 < 1/4c,$$

$$(c) \quad \lim_{t \rightarrow 0^+} \int_0^x u(\xi, t) d\xi = H(x) \quad \text{for all } x,$$

$$(d) \quad \int_0^x u(\xi, t) d\xi \leq Le^{Nx^2} \quad \text{on} \quad 0 < t \leq t_0 < 1/4c,$$

where $L = L(t_0)$ and $N = N(t_0)$.

PROOF. If $0 < t < 1/4c$, then by direct calculation,

$$\begin{aligned} u(x, t) &= \lim_{A \rightarrow \infty} \int_{-A}^A W_0(x - \xi, t) dH(\xi) \\ &= \lim_{B \rightarrow \infty} \left\{ [W_0(x - \xi, t) H(\xi)]_{-A}^A + \int_{-A}^A W_1(x - \xi, t) H(\xi) d\xi \right\}. \end{aligned}$$

By (4) and $|H(x)| \leq Me^{cx^2}$, the term in square brackets tends to zero. Hence

$$(8) \quad u(x, t) = \int_{-\infty}^{\infty} W_1(x - \xi, t) H(\xi) d\xi.$$

Lemma 1 assures us that (8) converges absolutely for $0 < t < 1/4c$, and has the properties (a), (b). Hence we can interpret $u(x, t)$ as coming from an initial dipole distribution of density $H(x)$.

To prove (c), integrate (8). Defining

$$(9) \quad H(x, t) = \int_0^x u(\xi, t) d\xi,$$

we get

$$(9') \quad H(x, t) = \int_{-\infty}^{\infty} [W_0(x - \xi, t) - W_0(-\xi, t)] H(\xi) d\xi,$$

where the Fubini theorem is used to interchange the order of integration. Letting t tend to zero, and using (γ) of Theorem A, we get (c). To prove (d), we apply (β) of Theorem A to both terms of $(9')$ and add.

To prove uniqueness, we shall want the following result.

THEOREM C. *If $u(x, t) \in C^2$, $u_{xx} = u_t$, and $|H(x, t)| \leq Ke^{Nx^2}$, where $H(x, t)$ is defined by (9) for all $0 \leq t \leq T$, then*

$$(10) \quad u(x, t) = \int_{-\infty}^{\infty} W_1(x - \xi, t) H(\xi, 0) d\xi, \quad \text{for } t < 1/4N.$$

PROOF. We first note that

$$H_t = \int_0^x u_t(\xi, t) d\xi = \int_0^x u_{xx}(\xi, t) d\xi = u_x(x, t) - u_x(0, t),$$

$$H_{xx} = u_x(x, t).$$

Hence, if we define $\bar{H}(x, t) = H(x, t) + \int_0^t u_x(0, \tau) d\tau$, then $\bar{H}(x, t)$ satisfies the heat equation, $\bar{H}_{xx} = \bar{H}_t$. Also $\bar{H}(x, t) \in C^2$ and $|\bar{H}(x, t)| \leq K'e^{Nx^2}$. Hence Theorem B applies, and

$$\bar{H}(x, t) = \int_{-\infty}^{\infty} W_0(x - \xi, t) \bar{H}(\xi, 0) d\xi = \int_{-\infty}^{\infty} W_0(x - \xi, t) H(\xi, 0) d\xi.$$

Using part (a) of Lemma 1, we differentiate with respect to x and obtain (10), q.e.d.

THEOREM 2. *Let $u(x, t) \in C^2$, $u_t = u_{xx}$, $|H(x, t)| \leq Ke^{Nx^2}$, where $H(x, t)$ is defined by (9), for all $0 < t \leq T$, and $H(x, t) \rightarrow 0$ as $t \rightarrow 0^+$ for almost all x . Then $u(x, t) \equiv 0$ on $0 < t \leq T$.*

PROOF. Let $0 < \tau < t < \min [1/4N, T]$, and $u_1(x, t - \tau) = u(x, t)$, $H_1(x, t - \tau) = H(x, t)$. Then for $t \geq \tau$, $u_1(x, t - \tau)$, $H_1(x, t - \tau)$ satisfy the conditions of Theorem C. Hence

$$u_1(x, t - \tau) = \int_{-\infty}^{\infty} W_x(x - \xi, t - \tau) H_1(\xi, 0) d\xi,$$

or

$$u(x, t) = \int_{-\infty}^{\infty} W_x(x - \xi, t - \tau) H(\xi, \tau) d\xi.$$

By hypothesis $H(\xi, \tau) \rightarrow 0$ almost everywhere *dominatedly* as $\tau \rightarrow 0$. Hence, using the Lebesgue dominated convergence theorem again, we have $u(x, t) \equiv 0$, $0 < t < \min [1/4N, T]$. If $1/4N < T$, we repeat the argument, considering $\bar{u}(x, t) = h(x, t + 1/4N)$ instead of $u(x, t)$: $\bar{u}(x, 0) = 0$ by the continuity of $u(x, t)$ in $0 < t < T$. This shows $u(x, t) \equiv 0$, $0 < t < \min [1/4N, T]$. The proof is complete after at most $[4NT]$ steps.²

COROLLARY. *Under the hypotheses of Theorem 1, $u(x, t)$ as defined by (6) is the only function satisfying conditions (a), (c), (d).*

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² J. L. B. Cooper (J. London Math. Soc. vol. 25 (1950) pp. 173-180) has proved a result related to Theorem 2. However, he requires that $\int_a^b g(x)u(x, t)dx \rightarrow 0$ for all bounded integrable $g(x)$; we require only $\int_a^b u(x, t)dx \rightarrow 0$, for all a, b . Cf. also J. Kampé de Fériet, C. R. Acad. Sci. Paris vol. 236 (1953) pp. 1527-1929.