IDENTITIES IN FINITE ALGEBRAS

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A finite algebraic system $\mathcal{A}$ will be constructed, with the property that the set of all laws of $\mathcal{A}$ is not a consequence of any finite subset of these laws.¹

$\mathcal{A}$ is of order seven, with a binary (non-associative) operation written as multiplication, and a 0-ary operation (or constant element) 0. Denoting the elements of $\mathcal{A}$ by 0, e , b_1 , b_2 , c , d_1 , d_2 , the multiplication is defined by setting

$$ce = c, \quad cb_j = d_j, \quad d_je = d_j, \quad d_jb_k = d_j \quad (j, k = 1, 2)$$

and setting all other products equal to 0.

LEMMA 1. *The following laws hold in $\mathcal{A}$:*

- (A123): $0x = 0, \quad x0 = 0, \quad x(yz) = 0;$
 (Bn): $((\cdots ((x_1x_2)x_3) \cdots)x_n)x_1 = 0 \quad (n = 1, 2, \cdots);$
 (Cn): $((\cdots ((x_1x_2)x_3) \cdots)x_n)x_2 = (\cdots ((x_1x_2)x_3) \cdots)x_n$
($n = 1, 2, \cdots$).

Moreover, every law of $\mathcal{A}$ that contains no more than n indeterminates is a consequence of A123 and of Bm, Cm for $m \leq n$.

PROOF. Verification that A123, Bn and Cn hold identically in $\mathcal{A}$ is routine. For the second assertion, let $\phi_\alpha = \phi_\beta$ be any law of $\mathcal{A}$ containing at most n indeterminates. By A we may suppose that each member is either 0 or a left-associated product, which we abbreviate as

$$\phi_\alpha = x_{\alpha_1}x_{\alpha_2} \cdots x_{\alpha_\mu}, \quad \phi_\beta = x_{\beta_1}x_{\beta_2} \cdots x_{\beta_\nu}.$$

By Bm and Cm for $m \leq n$, we can eliminate any repetition among the first $n+1$ arguments of ϕ_α or of ϕ_β , whence we may suppose that $\mu, \nu \leq n$ and that neither product contains repetitions. If ϕ_α , say, is not 0, then setting $x_{\alpha_1} = c$, $x_{\alpha_2} = \cdots = x_{\alpha_\mu} = e$, and any other $x = 0$, gives $\phi_\alpha = c$; it follows that ϕ_β is not 0, and indeed that $x_{\beta_1} = x_{\alpha_1}$, and that all the indeterminates of ϕ_β are among those of ϕ_α . By sym-

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metry, $\mu = \nu$ and the x_{α_i} are a permutation of the x_{β_i} . If some x_1 preceded x_2 in ϕ_α , while x_2 preceded x_1 in ϕ_β (therefore $x_{\alpha_1} \neq x_1, x_2$), then setting $x_{\alpha_1} = c, x_1 = b_1, x_2 = b_2$ and all other $x = e$ gives $\phi_\alpha = d_1$ and $\phi_\beta = d_2$. Since this contradicts $\phi_\alpha = \phi_\beta$, the x_{β_i} must be the identical permutation of the x_{α_i} , whence ϕ_α and ϕ_β are formally identical.

Define $\psi_\alpha = x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n}$ to have property P_n if ψ_α contains n indeterminates, and in the product ψ_α each of them occurs at least once to the left of the second occurrence of x_{α_1} . Note that this implies that if x_{α_k} is a later occurrence of x_{α_1} , then $k > n$.

LEMMA 2. *If $\psi_\alpha = \psi_\beta$ is a formal consequence of the laws A and Bm, Cm for $m < n$, and if ψ_α has the property P_n , then ψ_β also has the property P_n .*

PROOF. It is sufficient to consider the case where $\psi_\alpha = \psi_\beta$ results from a single application of A, Bm, or Cm. Now, no application of A is possible, since ψ_α contains no part 0 or of the form $u(vw)$. For an application of Bm or Cm we may suppose, by symmetry, that ψ_α contains a part of the form $y_1 y_2 \cdots y_m y_d$, where $y_d = y_1$ in the case of Bm and $y_d = y_2$ in the case of Cm. Clearly $y_1 = x_{\alpha_1} \cdots x_{\alpha_k}$ for some $k \geq 1$, and therefore $y_2 = x_{\alpha_{k+1}}, \cdots, y_d = x_{\alpha_{k+m}}$. For Bm, $y_d = y_1 = x_{\alpha_1} \cdots x_{\alpha_k}$ is clearly impossible for $k > 1$; while if $k = 1, y_d = x_{\alpha_{m+1}}$ is a later occurrence of $y_1 = x_{\alpha_1}$, impossible by property P_n since $m+1 \leq n$. Therefore no Bm, $m < n$, is applicable. Finally, if some Cm is applicable it can only delete an occurrence after the first of some indeterminate other than x_{α_1} , or else an occurrence after the second of x_{α_1} , and hence in either case preserves the property P_n .

THEOREM. *If L is any finite set of laws of $\mathcal{A}$, then some one of the laws Bn is not a consequence of L.*

PROOF. The finite set L of laws will contain fewer than n indeterminates, for some finite n . By Lemma 1, any consequence of L is a consequence of A123 and Bm, Cm for $m < n$. Now $x_1 x_2 \cdots x_n x_1$, the left member of Bn, has property P_n , while 0, the right member, has not. Therefore, by Lemma 2, Bn is not a consequence of A123 and Bm, Cm for $m < n$, whence Bn is not a consequence of L.

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