ON A CLASS OF FUNCTIONS SCHLICHT IN THE UNIT CIRCLE

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Introduction. Actual tests to determine whether a regular function is schlicht in $|z| < 1$ lead usually to tedious calculations of considerable difficulty. In this paper a class of polynomials having $|z| = 1$ as radius of schlichtness is investigated, for which a rather simple condition—both necessary and sufficient—in terms of the coefficients is given. With the aid of this condition it is also possible to obtain for this class of polynomials better results for certain quantities connected with the conformal mapping of schlicht functions, such as: bound of convexity (Rundungs Schranke), distance of the boundary in the w-plane from $w = 0$, etc.

Let S_p be the class of functions having $|z| = 1$ as radius of schlichtness, and let $f_p(z) = z - \sum_{n=2}^{N} a_n z^n$, having all a_n real and non-negative for $n = 2, 3, 4, \cdots, N$, $N \geqq 2$. Then we have:

THEOREM 1. *A necessary and sufficient condition for* $f_p(z) \in S_p$ is[1]

$$1 - \sum_{n=2}^{N} n a_n = 0.$$

PROOF. To show sufficiency, suppose there exist z_1, z_2, $|z_1|$, $|z_2| \leqq \rho < 1$, such that $f_p(z_1) = f_p(z_2)$. Then we have:

$$0 = f_p(z_1) - f_p(z_2) = z_1 - z_2 - \sum_{n=2}^{N} a_n(z_1^n - z_2^n)$$
$$= (z_1 - z_2)\left\{1 - \sum_{n=2}^{N} a_n(z_1^{n-1} + z_1^{n-2} z_2 + \cdots + z_2^{n-1})\right\}.$$

But

$$\left|1 - \sum_{n=2}^{N} a_n(z_1^{n-1} + z_1^{n-2} z_2 + \cdots + z_2^{n-1})\right|$$
$$\geqq 1 - \sum_{n=2}^{N} n a_n \rho^{n-1} > 1 - \sum_{n=2}^{N} n a_n = 0$$

and therefore $f_p(z_1) = f_p(z_2)$ implies $z_1 = z_2$. To show that the condition is also necessary, we have: A necessary condition for a function to be

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[1] The sufficiency of this condition is a well known result. It is proved here for the sake of completeness.

schlicht in $|z|<1$ is that $f'(z)\neq 0$ for $|z|<1$. But

$$|f_p'(z)| = \left|1-\sum_{n=2}^{N} na_n z^{n-1}\right| \geqq 1-\sum_{n=2}^{N} na_n |z|^{n-1} > 1-\sum_{n=2}^{N} na_n.$$

Now, suppose $1-\sum_{n=2}^{N} na_n<0$, then there exists a real $z=z_0$ such that $1-\sum_{n=2}^{N} na_n z_0^{n-1}=f_p'(z_0)=0$ since $f_p'(0)=1$ and $f_p'(1)<0$. Therefore, we must have $1-\sum_{n=2}^{N} na_n>0$. But if $1-\sum_{n=2}^{N} na_n>0$ then we would have schlichtness for $|z|=R>1$. Hence the condition is necessary.

Corollary. *For functions $f_p(z)$ of class S_p we have $a_k\leqq 1/k$, $k=2, 3, \cdots, N$. If $a_m=1/m$, then $f_p(z)=z-(1/m)z^m$.*

Theorem 2. *If $w=f_p(z)\in S_p$, then the map of $|z|\leqq 1$ cannot be a convex region.*

Proof. A necessary and sufficient condition that $w=f(z)$ should map $|z|\leqq 1$ into a convex region is $\operatorname{Re}\{zf''(z)/f'(z)+1\}\geqq 0$ for all $|z|\leqq 1$. For $f_p(z)\in S_p$ this condition becomes

$$\operatorname{Re}\left\{\frac{1-\sum_{n=2}^{N} n^2 a_n z^{n-1}}{1-\sum_{n=2}^{N} na_n z^{n-1}}\right\}\geqq 0.$$

Now, considering the numerator of this expression we notice that for $z=0$ its value is positive and for $z=1$ its value is negative, since $1-\sum_{n=2}^{N} n^2a_n<1-\sum_{n=2}^{N} na_n=0$. Therefore, there exists a real $z=z_0$ such that the numerator is equal to zero and will become and stay negative in an interval between $z=z_0$ and $z=1$, while the denominator stays positive (by Theorem 1), and therefore

$$\operatorname{Re}\left\{\frac{1-\sum_{n=2}^{N} n^2 a_n z^{n-1}}{1-\sum_{n=2}^{N} na_n z^{n-1}}\right\}< 0 \qquad \text{for } z_0<z\leqq 1,$$

which proves the theorem. However, we have:

Theorem 3. *All $w=f_p(z)\in S_p$ map $|z|\leqq 1$ into a region star-shaped with respect to $w=0$.*

Proof. A necessary and sufficient condition for $w=f(z)$ to map $|z|\leqq 1$ into a star-shaped region with respect to $w=0$ is

Re $\{zf'(z)/f(z)\} \geqq 0$ for all $|z| \leqq 1$. In our case

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{\frac{1-\sum_{n=2}^{N} na_n z^{n-1}}{1-\sum_{n=2}^{N} a_n z^{n-1}}\right\} = \operatorname{Re}\left\{1-\sum_{n=1}^{\infty} b_n z^n\right\}$$

where $b_1 = a_2$ and $b_n = na_{n+1} + \sum_{k=1}^{n-1} b_k a_{n-k+1}$ for $n = 2, 3, 4, \cdots$. It follows therefore by induction that all b_n are positive. Therefore:

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{1-\sum_{n=1}^{\infty} b_n z^n\right\} \geqq \operatorname{Re}\left\{1-\sum_{n=1}^{\infty} b_n |z|^n\right\}$$

$$\geqq 1 - \sum_{n=1}^{\infty} b_n = 0.$$

(Note: The last equality follows from the fact that the numerator of Re $\{zf'(z)/f(z)\} = 0$ for $z = 1$.) But $zf'(z)/f(z)$ is regular for $|z| \leqq 1$, and therefore Re $\{zf'(z)/f(z)\}$ is a harmonic function, which satisfies Re $\{zf'(z)/f(z)\} \geqq 0$ on $|z| = 1$. But a harmonic function cannot take a minimum "inside" and therefore Re $\{zf'(z)/f(z)\} \geqq 0$ for all $|z| \leqq 1$. This completes the proof.

THEOREM 4. *If $w = f_p(z) \in S_p$ and d^* is a point in the w-plane such that $f_p(z) \neq d^*$, $|z| \leqq 1$, then $d^* \geqq 1/2$, i.e. the circle $|w| \leqq 1/2$ is always covered by the map of the unit circle by $f_p(z) = S_p$.*

PROOF.

$$|f(e^{i\theta})| = \left|e^{i\theta} - \sum_{n=2}^{N} a_n e^{in\theta}\right| \geqq 1 - \sum_{n=2}^{N} a_n \geqq 1 - \frac{1}{2}\sum_{n=2}^{N} na_n = \frac{1}{2}\cdot$$

This inequality is sharp, since for $f_p(z) = z - z^2/2 \in S_p$ we have $f(1) = 1/2$. We also note that if $f_p(z) \in S_p$ is of degree k, then $d^* \leqq 1 - 1/k$ since $a_2 + a_3 + \cdots + a_k \geqq (1/k) \sum_{n=2}^{k} na_n = 1/k$.

THEOREM 5. *The bound of convexity of $w = f_p(z) \in S_p$ is 1/2, i.e. $|z| = r_0 \leqq 1/2$ is always mapped into a convex curve, but not always $|z| = r > 1/2$.*

PROOF. The bound of convexity is the least zero—in absolute value—of $\operatorname{Re}\{zf''(z)/f'(z) + 1\} = \operatorname{Re}\{(1 - \sum_{n=2}^{N} n^2 a_n z^{n-1})/(1 - \sum_{n=2}^{\infty} na_n z^{n-1})\} = 1 - \sum_{n=1}^{\infty} c_n z^n$ where all $c_n \geqq 0$. By considerations similar to those of Theorem 3, we find that the least zero will be a positive real number $0<z_0<1$. We must have, therefore, a z_0 such that $1-\sum_{n=2}^{N} n^2a_nz_0^{n-1}=0$.

Suppose now $z_0<1/2$; then $1-\sum_{n=2}^{N} n^2a_n(1/2)^{n-1}<0$. But

$$1-4a_2\cdot\frac{1}{2}-9a_3\cdot\frac{1}{4}-\cdots-N^2a_N\frac{1}{2^{N-1}}$$

$$=1-2a_2-\frac{9}{4}a_3-\frac{N^2a_N}{2^{N-1}}>1-2a_2-3a_3-\cdots-Na_N=0$$

which is a contradiction. The estimate is exact since for $f_p(z)=z-z^2/2$ we have $r_0=1/2$.

THEOREM 6. *The circle $|w|\leqq 3/8$ is always covered by a convex region, i.e. if d_0 is the shortest distance from $w=0$ to $f_p(r_0i^{i\theta})$, where r_0 is the bound of convexity, then $d_0\geqq 3/8$.*

PROOF. $d_0=|f(r_0e^{i\theta})|=|r_0e^{i\theta}-\sum_{n=2}^{N} a_nr_0^ne^{in\theta}|\geqq r_0-\sum_{n=2}^{N} a_nr_0^n=r_0\{1-\sum_{n=2}^{N} a_nr_0^{n-1}\}\geqq r_0\{1-r_0\sum_{n=2}^{N} a_n\}\geqq r_0\{1-r_0/2\}$. We know by the previous theorem that $r_0\geqq 1/2$. But $r_0\{1-r_0/2\}$ is strictly increasing in $1/2\leqq r_0<1$, therefore

$$d_0\geqq\frac{1}{2}\left\{1-\frac{1/2}{2}\right\}=\frac{3}{8}\quad\text{q.e.d.}$$

For the function $f(z)=z-z^2/2$ this estimate is exact. By Theorems 4 and 6 we have

$$1/2\leqq d^*<1,\qquad 3/8\leqq d_0<d^*$$

and therefore obviously

$$d_0/d^*>3/8.$$

It has been conjectured that for all functions $w=f(z)=z+\sum_{n=2}^{\infty} a_nz^n$ regular and schlicht in the unit circle which map $|z|\leqq 1$ into a star-shaped region we have $d_0/d^*\geqq 2/3$. (This lower bound for d_0/d^* cannot be improved since, for $f(z)=z(1+z)^{-2}$, $d^*=1/4$, $d_0=1/6$.) It is possible to prove this conjecture for all $f(z)\in S_p$.

THEOREM 7. *For all $f_p(z)\in S_p$ we have $d_0/d^*\geqq 2/3$.*

PROOF.

$$d_0=\int_0^{z_0} f_p'(z)dz\geqq z_0f_p'(z_0) \tag{1}$$

since $f_p'(z)$ is decreasing for $0 \leqq z \leqq z_0$, Im $z=0$. Also

$$d^* = \int_0^1 f_p'(z)dz = \int_0^{z_0} f_p'(z)dz + \int_{z_0}^1 f_p'(z)dz = d_0 + \int_{z_0}^1 f_p'(z)dz.$$

But for all $f_p(z) \in S_p$, $f_p'''(z) \leqq 0$ for $0 \leqq z \leqq 1$, Im $z=0$, i.e. $f_p'(z)$ is convex upward in this interval and therefore the tangent to $f_p'(z)$ at $(z_0, f'(z_0))$ will lie entirely above the $f_p'(z)$ curve. But at $(z_0, f'(z_0))$ we have $z_0 f''(z_0)/f'(z_0)+1=0$, i.e. $f''(z_0) = -f'(z_0)/z_0$, i.e. the slope of the tangent at $(z_0, f'(z_0))$ is $= -f'(z_0)/z_0$, and therefore the tangent will intersect the Re z-axis at $z=2z_0$. Therefore $\int_{z_0}^1 f_p'(z)dz \leqq$ area of the triangle formed by the three points:

$$(z_0, f'(z_0)), \qquad (z_0, 0), \qquad (2z_0, 0),$$

i.e. $\int_{z_0}^1 f_p'(z)d_z \leqq z_0 f_p'(z_0)/2$. Thus $d^* \leqq d_0 + z_0 f_p'(z_0)/2$ and on account of (1)

$$d^* \leqq d_0 + d_0/2 = 3\, d_0/2, \quad \text{i.e.} \quad d_0/d^* \geqq 2/3.$$

It seems highly probable that, for $f_p(z) \in S_p$, we actually have $d_0/d^* \geqq 3/4$, which could not be improved, since it is sharp for the function $f_p(z) = z - z^2/2$.

THEOREM 8. *For the area of the map of* $|z| \leqq 1$ *by* $f_p(z) \in S_p$, *we have*

$$\pi < A \leqq 3\pi/2.$$

PROOF. The left inequality is obvious, since $A = \pi\{1 + \sum_{n=2}^N na_n^2\}$. To prove the right-hand side of the inequality, we have:

$$\pi\left\{1 + \sum_{n=2}^N na_n^2\right\} \leqq \pi\left\{1 + \sum_{n=2}^N n \cdot \frac{1}{n} \cdot a_n\right\} = \pi\left\{1 + \sum_{n=2}^N a_n\right\}$$
$$\leqq \pi\left\{1 + \frac{1}{2}\right\} = \frac{3}{2}\pi.$$

This is exact, since for $f_p(z) = z - z^2/2$, $A = 3\pi/2$.

THEOREM 9. *For the functions* $f_p(z)$ *of class* S_p *we have the following distortion theorem*:

$$|z| - |z|^2/2 \leqq |f_p(z)| \leqq |z| + |z|^2/2.$$

PROOF. $|f(z)| = |z - \sum_{n=2}^N a_n z^n| \leqq |z| + \sum_{n=2}^N a_n |z|^n \leqq |z| + |z|^2 \cdot \sum_{n=2}^N a_n \leqq |z| + |z|^2/2$.

Similarly, $|f_p(z)| \geqq |z| - \sum_{n=2}^N a_n |z|^n \geqq |z| - |z|^2 \sum_{n=2}^N a_n \geqq |z| - |z|^2/2$.

THEOREM 10. *For the derivatives of functions $f_p(z)$ of class S:*

$$1-|z| \leqq |f_p'(z)| \leqq 1+|z|.$$

These inequalities are sharp. They are attained by $f_p(z)=z-z^2/2$ at $z=\pm r$, r real.

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A REMARK ON REVERSIBLE MATRICES

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The matrix $A=(a_{nk})$ is called *reversible* if for each $y=\{y_n\}\in(c)$ the equations $y_n=\sum_{k=0}^{\infty} a_{nk}x_k$ $(n=0, 1, \cdots)$ have exactly one solution $x=\{x_k\}$. In this case there exist [1; 4] constants c_k, b_{kn} with $\sum_n |b_{kn}|<\infty$ for each k, such that

$$x_k = c_k \lim_{n\to\infty} y_n + \sum_{n=0}^{\infty} b_{kn}y_n \qquad (k=0, 1, \cdots). \tag{1}$$

It is further stated in [1, p. 50] that the c_k are bounded. This is questioned in [4], where it is pointed out that if the c_k were generally bounded they would have to be almost all zero, but this remark does not dispose of the matter, for it might conceivably be a true theorem that for each reversible matrix the c_k are almost all zero. (For row-finite matrices, all c_k vanish; [3].) The example given in [4, p. 47] seems inconclusive. The purpose of this note is to show by a very simple example that in fact the c_k need not be bounded.

Consider the transformation

$$y_{2m} = \sum_{p=0}^{m} x_{2p}, \qquad y_{2m+1} = 2^{-m}x_{2m+1} + \sum_{p=0}^{\infty} x_{2p},$$

where $m=0, 1, \cdots$. For each $y\in(c)$ we have

$$x_{2m+1} = 2^m(y_{2m+1} - \lim_n y_n);$$

thus $c_{2m+1}=-2^m$ is not bounded.

This has a bearing on a paper [2] in which the following theorem is stated:

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