ON THE ORDER OF THE AUTOMORPHISM GROUP OF A FINITE GROUP

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Let G be a finite group, $A(G)$ its group of automorphisms, and p a prime. Several authors have given upper bounds on the order $o(A(G))$ of $A(G)$ in terms of $o(G)$. Birkhoff and Hall [1, p. 499] suggested the problem of determining a lower bound for $o(A(G))$ in terms of $o(G)$. For Abelian groups, this had already been done by Hilton [3]. He proved that if $p^n | o(G)$, then $p^{n-1}(p-1) | o(A(G))$ for Abelian groups G . The only result obtained in the general case is due to Herstein and Adney [2]. They proved that if $p^2 | o(G)$, then $p | o(A(G))$.

In this note, it is shown that if $p^3 | o(G)$, then $p^2 | o(A(G))$. One is tempted to conjecture that there is a function $f(n)$ such that if $p^{f(n)} | o(G)$, then $p^n | o(A(G))$. However the methods used here will not suffice to prove even the case $n=3$.

Let $I(G)$ be the group of inner automorphisms of G . Let the operation in G be addition. The union of subgroups H and K will be denoted by $H \cup K$, the direct sum by $H + K$, the factor group by H/K , and the point set difference by $H - K$. Let $[H:K]$ denote the index of K in H .

LEMMA 1. *If G is Abelian and $p^n | o(G)$, then $p^{n-1} | o(A(G))$.*

This was proved in [3].

LEMMA 2. *Let $G = S \cup M$ where S is a subgroup and M a normal subgroup of G , and where $S \cap M = 0$. Let $Z(G)$ be the center of G . Let α' be an automorphism of S such that $s\alpha' - s \in Z(G)$ for all $s \in S$. Let $(s+m)\alpha = (s\alpha') + m$ for $s \in S$, $m \in M$. Then α is an automorphism of G .*

LEMMA 3.¹ *If S is an Abelian group of order p^n , $n > 1$, and Z is a subgroup of index p , then there is a group $U \subseteq A(S)$ with $o(U) = p^{n-2}$ such that $s\alpha - s \in Z$ for all $s \in S$ and all $\alpha \in U$.*

PROOF. Let w be an element of smallest order in $S - Z$. If $p^r w = p^r z$, then $p^r(w - z) = 0$. It follows that $\{p^r w\}$ is pure in Z . Hence $Z = \{p^r w\} + T$ and $S = \{w\} + T$, $T \subseteq Z$. Let $o(w) = p^i$, $o(T) = p^j$. Let V' be the group of automorphisms α of $\{w\}$ such that $w\alpha = (1 + \lambda p)w$. Then

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¹ Lemma 3 is due to the referee. It permitted the proof of the present Theorem 2, which was the only theorem proved in the original version of the paper, to be used as a proof of Theorem 1 with only minor changes.

$o(V') = p^{i-1}$. Let W' be a subgroup of $A(T)$ of order p^{i-1} whose existence is guaranteed by Lemma 1. There are subgroups V and W of $A(S)$ corresponding in an obvious way to V' and W' . The group $U = \{V, W\}$ has the required properties (if $\{pw\}$ or T is 0, then V or W , respectively, is defined to be the identity).

THEOREM 1. *Let G be a finite group, S a Sylow p -subgroup of order p^n , and Z the intersection of S with the center of G . If $[S:Z] \geq p^{n-1}$, $[S:Z] = 1$, or $[S:Z] = p$, then $p^{n-1} \mid o(A(G))$.*

PROOF. Case 1. $[S:Z] \geq p^{n-1}$. Then $p^{n-1} \mid o(I(G)) \mid o(A(G))$.

Case 2. $[S:Z] = 1$. Let σ be the transfer of G into S . If $z \in Z$, then $z\sigma = [G:S]z$, hence $Z\sigma = Z$. Therefore, if N is the kernel of σ , $G = Z + N$. By Lemma 1, $p^{n-1} \mid o(A(Z)) \mid o(A(G))$.

Case 3. $[S:Z] = p$. Since the center $Z(S)$ of S contains Z , and since $S/Z(S)$ is not cyclic of order p , S is Abelian. Let σ be the transfer of G into S . Then $Z\sigma = Z$. If $G\sigma = Z$, then, as in case 2, $G = Z + N$. By Lemma 1, there is a group $U' \subseteq A(Z)$ with $o(U') = p^{n-2}$. Let U be the corresponding subgroup of $A(G)$. Since $[S:Z] = p$, an element $s_0 \in S - Z$ induces an automorphism ρ of order p . If $\alpha \in U$, then $\alpha\rho = \rho\alpha$, and $\rho \notin U$. Therefore $\{\rho, U\}$ is a subgroup of $A(G)$ of order p^{n-1} .

Next, assume that $G\sigma = S$. If M is the kernel of σ , then $o(M) = [G:S]$, and therefore $S \cap M = 0$, $S \cup M = G$, and M is normal in G . By Lemma 3, there is a group U' of automorphisms of S with $o(U') = p^{n-2}$ such that $s\alpha' - s \in Z$ for all $s \in S$ and all $\alpha' \in U'$. By Lemma 2, there is a corresponding group U of automorphisms of G . Let ρ be defined as before, let $\alpha' \in U'$, and let α be the corresponding element of U (see Lemma 2). Then

$$(s + m)\alpha\rho = s\alpha' + m\rho = (s + m)\rho\alpha.$$

Since ρ induces the identity automorphism on S , $\rho \notin U$, and $\{\rho, U\}$ is a subgroup of $A(G)$ of order p^{n-1} .

THEOREM 2. *If $p^3 \mid o(G)$, then $p^2 \mid o(A(G))$.*

PROOF. This follows immediately from Theorem 1, and the remark made in Case 1 of its proof.

BIBLIOGRAPHY

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