LOEWY CHAINS AND UNIFORM SPLITTING OF LATTICES

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1. Introduction. This paper is devoted to a proof of a theorem on the existence of a Loewy η -chain as a sufficient condition for uniform splitting. The system considered is a complete modular lattice L (with the usual binary operations denoted by $p+q$ and pq for p and q in L) satisfying the additional Property (*): $a \sum_{\alpha} p_{\alpha} = \sum_{\alpha} ap_{\alpha}$ for a and the ascending chain p_{α} elements of L .

For a complete statement of results made use of here, the reader is referred to a previous paper [1]. For convenience some basic definitions from that paper are listed in the remainder of this section. A single-valued transformation η is a *homomorphism* of p/p' in L upon q/q' in K (a system satisfying the same postulates as L , or L itself) if η maps x/p' for every x satisfying $p' \leq x \leq p$ upon $x\eta/q'$, where $q' \leq x\eta \leq q$, so that the following requirements hold:

- (1) $(\sum_x x)\eta = \sum_x (x\eta)$.
- (2) If $q' \leq z \leq x\eta$, then there exists an element z' with $p' \leq z' \leq x$ such that $z'\eta = z$.
- (3) If $x\eta = y\eta$, then there exist elements x' and y' ($p' \leq x' \leq p$ and $p' \leq y' \leq p$) such that $x + x' = y + y'$ and $x'\eta = y'\eta = q'$.

A homomorphism η of $p/0$ upon $p\eta/0$ is said to be an *endomorphism* of $p/0$ if $p\eta \leq p$. The *kernel* $k(\eta)$ of this endomorphism is the sum of all elements which η maps upon 0, and the *radical* $r(\eta)$ is defined as $\sum_{i=1}^{\infty} k(\eta^i)$. x is η -*admissible* if $x\eta \leq x$. The usual homomorphism and isomorphism theorems hold.

The element c is a *complement* of the endomorphism η of $p/0$ if $p = r(\eta) + c$, $cr(\eta) = 0$ (that is, $p = c \oplus r(\eta)$), and $c\eta = c$. If there exists a complement of η , then η is a *splitting endomorphism* of $p/0$. If η induces a splitting endomorphism of $x/0$ for every η -admissible x , η is a *uniformly splitting endomorphism* of $p/0$.

2. Proof of the theorem. Several definitions and lemmas are required for the proof of the following theorem.

THEOREM. *The endomorphism η of $p/0$ is a uniformly splitting endomorphism if there exists a Loewy η -chain the sum of which is p .*

DEFINITION 1. An η -admissible element q is *minimal η -admissible over r* if $r < s \leq q$, for η -admissible s , implies that $q = s$.

DEFINITION 2. Elements a_i ($i = 0, 1, 2, \dots$) form a *Loewy η -chain*

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if $a_0 = 0$ and every a_i with $i > 0$ is the sum of elements which are minimal η -admissible over a_{i-1} .

It is clear from the definition that the chain of the a_i is ascending and that every a_i is η -admissible.

LEMMA 1. *Let η be a homomorphism of m/r upon $m\eta/r$ and let m be minimal η -admissible over r . Then either $m\eta = r$ or $m\eta$ is minimal η -admissible over r . Furthermore, if m is a sum of elements minimal η -admissible over r , then $m\eta = r$ or $m\eta$ is a sum of elements minimal η -admissible over r .*

PROOF. Suppose that $m\eta \neq r$ and that there exists an η -admissible s such that $r < s < m\eta$. Since $m\eta \leq m$, the second supposition contradicts the fact that m is minimal η -admissible. Thus $m\eta$ is minimal η -admissible over r . Moreover, if $m = \sum_{\alpha} m_{\alpha}$, $(\sum_{\alpha} m_{\alpha})\eta = \sum_{\alpha} (m_{\alpha}\eta)$, and the preceding result gives the second conclusion.

LEMMA 2. *If $z \leq y \leq x$, if x is the sum of elements minimal η -admissible over z (and included in x), and if $y\eta \leq y$, then there exists t^* such that $x = y + t^*$ and $yt^* = z$, and y is equal to z or is the sum of elements minimal η -admissible over z (and included in y).*

PROOF. Let $A = \{t \in L \mid t\eta \leq t, t \leq x, \text{ and } ty = z\}$, let $B = \{t_{\alpha}\}$ be a simply ordered subset of A , and let $t' = \sum_{\alpha} t_{\alpha}$. Then $t'\eta \leq t'$, $t_{\alpha} \leq t' \leq x$ for every α and $yt' = y \sum_{\alpha} t_{\alpha} = \sum_{\alpha} yt_{\alpha} = z$, by Property (*) and the definition of A . Thus the maximal principle is applicable and asserts the existence of a maximal element t^* in A , with $yt^* = z$.

Let $x = \sum_m m$, the sum of elements minimal η -admissible over z , and consider any m . Then $z \leq m(y + t^*) \leq m$, and hence either $m = m(y + t^*)$ or $m(y + t^*) = z$. If the latter were true, by the modular law, $y(t^* + m) = y(t^* + m)(y + t^*) \leq (t^* + m)(y + t^*) = t^* + m(y + t^*) = t^* + z = t^*$. Hence $z \leq y(t^* + m) \leq yt^* = z$, and $t^* + m$ is in A . By the maximal property of t^* , $t^* \leq t^* + m$, and hence $t^* = t^* + m$ and $m \leq t^*$. But then $m = mt^* = z$. In this case $x = \sum_m m = z$, and thus $y = x$, $t^* = z$, and the lemma is proved.

If $m = m(y + t^*)$, $m \leq y + t^*$. Consequently $x = \sum_m m \leq y + t^* \leq x$, or $x = y + t^*$. The last equation implies the existence of an isomorphism [1, p. 334, Theorem 3] of x/t^* upon y/z . Hence, by an obvious generalization of [1, p. 333, Theorem 1], there is a homomorphism of x/z upon y/z . Thus, by Lemma 1, y is the sum of elements minimal η -admissible over z (or z).

LEMMA 3. *If elements $a_i \leq p$ form a Loewy η -chain, then $a_i\eta(\eta) = a_i k(\eta^i)$ for all i .*

PROOF. The lemma is proved by induction with respect to i . Since $a_0 = 0$, the assertion is true for subscript 0. As induction hypothesis, it will be assumed that $a_{i-1}r(\eta) = a_{i-1}k(\eta^{i-1})$.

Let Q be the set of elements q , with $a_{i-1} < q \leq a_{i-1} + a_i r(\eta) = \sum_{n=1}^{\infty} (a_{i-1} + a_i k(\eta^n))$, which are minimal η -admissible over a_{i-1} . By Property (*)

$$q = q \sum_{n=1}^{\infty} (a_{i-1} + a_i k(\eta^n)) = \sum_{n=1}^{\infty} q(a_{i-1} + a_i k(\eta^n)).$$

Furthermore, the modular law shows that $q \geq q(a_{i-1} + a_i k(\eta^n)) = a_{i-1} + qa_i k(\eta^n) = a_{i-1} + qk(\eta^n) = q(a_{i-1} + k(\eta^n)) \geq a_{i-1}$. Thus

$$q = \sum_{n=1}^{\infty} q(a_{i-1} + k(\eta^n)) \quad \text{and} \quad a_{i-1} \leq q(a_{i-1} + k(\eta^n)) \leq q.$$

If $q(a_{i-1} + k(\eta^n)) = a_{i-1}$ for every n , then $q = \sum_{n=1}^{\infty} q(a_{i-1} + k(\eta^n)) = a_{i-1}$. Hence there exists an $m = m(q)$ such that $a_{i-1} < q(a_{i-1} + k(\eta^m)) \leq q$. Since q is minimal η -admissible over a_{i-1} , $q = q(a_{i-1} + k(\eta^m)) = a_{i-1} + qk(\eta^m)$. Thus $q\eta^m = a_{i-1}\eta^m \leq a_{i-1}$.

Since $a_{i-1} \leq a_{i-1} + q\eta \leq q$ and since $a_{i-1} + q\eta$ is η -admissible, either $a_{i-1} = a_{i-1} + q\eta$ or $q = a_{i-1} + q\eta$. If $q = a_{i-1} + q\eta$, then

$$a_{i-1} < q = a_{i-1} + q\eta = \dots = a_{i-1} + q\eta^m = a_{i-1},$$

since $q = a_{i-1} + (a_{i-1} + q\eta)\eta = a_{i-1} + a_{i-1}\eta + q\eta^2 = a_{i-1} + q\eta^2 = \dots$ and since $q\eta^m \leq a_{i-1}$. This impossibility shows that the other alternative, $a_{i-1} = a_{i-1} + q\eta$, must hold. Therefore $q\eta \leq a_{i-1}$ for every q .

By Lemma 2, $a_{i-1} + a_i r(\eta) = \sum_q q$, the sum of elements minimal η -admissible over a_{i-1} . Since every $q\eta \leq a_{i-1}$,

$$\sum_q (q\eta) = \left(\sum_q q \right) \eta = (a_{i-1} + a_i r(\eta)) \eta \leq a_{i-1}.$$

Moreover, $(a_i r(\eta))\eta \leq (a_{i-1} + a_i r(\eta))\eta \leq a_{i-1}$, and $(a_i r(\eta))\eta \leq a_{i-1}r(\eta) = a_{i-1}k(\eta^{i-1}) \leq k(\eta^{i-1})$, by the induction hypothesis. Therefore $[(a_i r(\eta))\eta]\eta^{i-1} = (a_i r(\eta))\eta^i = 0$, and this fact implies that $a_i r(\eta) \leq k(\eta^i)$. Hence $a_i k(\eta^i) = a_i r(\eta)$, and the lemma is proved.

LEMMA 4. If the a_i form a Loewy η -chain, then $a_i = a_j \eta^i + a_i r(\eta)$ for every i and j .

PROOF. The proof is by induction with respect to i . The assertion is trivially true for subscript 0. Assume, as induction hypothesis, that for $0 < i$, $a_{i-1} = a_{i-1}\eta^j + a_{i-1}r(\eta)$ for every j .

Let q be minimal η -admissible over a_{i-1} such that $a_{i-1} < q \leq a_i$.

Two cases are to be considered.

Case 1. $q\eta \leq a_{i-1}$. By Lemma 3 and the induction hypothesis

$$(1) \quad q\eta \leq a_{i-1} = a_{i-1}\eta + a_{i-1}k(\eta^{i-1}) = a_{i-1}\eta^i + a_{i-1}k(\eta^{i-1}).$$

Since $a_{i-1} < q$, it follows that $q\eta^i = a_{i-1}\eta^i$. It is a consequence of (1) also that $a_{i-1}\eta^i = a_{i-1}\eta^{j+i}$ for all j and hence $q\eta^i = (a_{i-1}\eta^j)\eta^i$ for all j . Since η^i is an endomorphism of $p/0$, the last equality shows that $q + k(\eta^i) = a_{i-1}\eta^j + k(\eta^i)$. Thus

$$q = qa_i \leq (a_{i-1}\eta^j + k(\eta^i))a_i = a_{i-1}\eta^j + a_i k(\eta^i) \leq a_i\eta^j + a_i r(\eta) \leq a_i.$$

Case 2. $q\eta \not\leq a_{i-1}$. In this case $a_{i-1} < a_{i-1} + q\eta \leq q$, and $a_{i-1} + q\eta$ is η -admissible. Hence, by the minimal property of q , $q = a_{i-1} + q\eta$. The induction hypothesis then implies that, for all j ,

$$\begin{aligned} q &= a_{i-1} + q\eta = a_{i-1} + q\eta^2 = \cdots = a_{i-1} + q\eta^j \\ &= a_{i-1}\eta^j + a_{i-1}r(\eta) + q\eta^j \\ &\leq a_i\eta^j + a_i r(\eta) + q\eta^j = a_i\eta^j + a_i r(\eta) \leq a_i. \end{aligned}$$

Since $a_i = \sum_q q$, in both cases $a_i = a_i\eta^j + a_i r(\eta)$ for every i and j .

LEMMA 5. *If the a_i form a Loewy η -chain, then $a_i = a_i\eta^i \oplus a_i r(\eta)$ and $a_i\eta^i = a_i\eta^{i+1}$ for every i .*

PROOF. By Lemma 4, $a_i = a_i\eta^j + a_i r(\eta)$ for all i and j . Hence for $j = 1$, by Lemma 3, $a_i = a_i\eta + a_i k(\eta^i)$, for all i . As a result, $a_i\eta^i = a_i\eta^{i+1}$, and the chain

$$a_i \geq a_i\eta \geq \cdots \geq a_i\eta^i = a_i\eta^{i+1} = \cdots$$

is finite. By Lemma 3, the chain

$$a_i k(\eta) \leq a_i k(\eta^2) \leq \cdots \leq a_i k(\eta^i) = a_i k(\eta^{i+1}) = \cdots = a_i r(\eta)$$

is also finite, where the terms are the kernels of the respective endomorphisms induced in a_i . [1, p. 339, Theorem 3] now gives the result $a_i = a_i\eta^i \oplus a_i r(\eta)$, for $i = 0, 1, 2, \dots$.

LEMMA 6. *An endomorphism η of $p/0$ is a splitting endomorphism of $p/0$ if there exists a Loewy η -chain the sum of which is p .*

PROOF. Let $a_0 \leq a_1 \leq \cdots$ be the Loewy η -chain. By Lemma 5, for each i , $a_i = a_i\eta^i \oplus a_i r(\eta)$. By hypothesis, it follows that

$$p = \sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} [a_i\eta^i + a_i r(\eta)] = \sum_{i=0}^{\infty} a_i\eta^i + \sum_{i=0}^{\infty} a_i r(\eta).$$

Let $\sum_{i=0}^{\infty} a_i\eta^i = c$. By Property (*),

$$\sum_{i=0}^{\infty} a_i r(\eta) = r(\eta) \sum_{i=0}^{\infty} a_i = r(\eta) p = r(\eta).$$

Hence $p = c + r(\eta)$.

By Lemma 5, $(a_i \eta^i) \eta = a_i \eta^i$ and $0 = (a_i \eta^i)(a_i r(\eta)) = (a_i \eta^i) r(\eta)$ for all i . Hence $c \eta = c$, and

$$c r(\eta) = r(\eta) \sum_{i=0}^{\infty} a_i \eta^i = \sum_{i=0}^{\infty} r(\eta) a_i \eta^i = 0.$$

Thus there exists a complement c , and η is a splitting endomorphism of $p/0$.

PROOF OF THE THEOREM. Let $a_0 \leq a_1 \leq \dots$ be the Loewy η -chain and let s be η -admissible and included in p . Then the sa_i form an ascending chain of η -admissible elements, and, by Property (*),

$$\sum_{i=0}^{\infty} sa_i = s \sum_{i=0}^{\infty} a_i = sp = s.$$

Since $a_{i-1} \leq a_{i-1} + sa_i \leq a_i = \sum_q q$, where q is minimal η -admissible over a_{i-1} , it follows by use of Lemma 2 that $a_{i-1} + sa_i$ is the sum of elements minimal η -admissible over a_{i-1} . A basic isomorphism theorem gives the existence of an isomorphism of $(a_{i-1} + sa_i)/a_{i-1}$ onto $sa_i/(sa_i)(a_{i-1}) = sa_i/sa_{i-1}$. Thus it can be concluded that sa_i is the sum of elements which are minimal η -admissible over sa_{i-1} . Thus the sa_i form an ascending Loewy η -chain, the sum of which is s , and, by Lemma 6, the endomorphism induced by η in $s/0$ is a splitting endomorphism of $s/0$. Hence η is a uniformly splitting endomorphism of $p/0$.

REFERENCE

1. L. A. Hostinsky, *Endomorphisms of lattices*, Duke Math. J. vol. 18 (1951) pp. 331-342.

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