ON A DOUBLE EIGENVALUE PROBLEM

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In the study of the rigidity of surfaces of revolution one encounters the following "double" eigenvalue problem [2; 3]: Find those numbers $a > 0$ for which there exists a not identically vanishing function y and an integer $m > 1$ such that y is a solution of the system

$$(1) \quad \left(\frac{y'}{p(x)} \right)' + \lambda \frac{q(x)}{(a + r(x))p^2(x)} y = 0, \quad y(0) = y(b) = 0,$$

where $\lambda = m^2 - 1$. Rembs [3] demonstrated the existence of a countably infinite set of such a 's under the assumption that the functions p , q , and r are analytic on the interval $[0, b]$, that $q > 0$ and $r \geq 0$ on the interval, and that $p > 0$ on the open interval, but vanishes to the first order at each end. Minagawa and Rado, in a more general setting [2], showed that these "exceptional a 's" are at most countable in number. It is the purpose of the present note to show that the a 's form a dense set on the positive real axis. For the validity of the following argument it is sufficient to assume that the functions p , q , and r are of class C'' rather than analytic and that p vanishes, if at all, to a lower order than the second at the end points.

For a fixed positive number a , the problem (1) is a (regular singular) Sturm-Liouville eigenvalue problem. It is known that the eigenvalues $\lambda_n(a)$ are positive and can be arranged in the sequence $0 < \lambda_1(a) < \lambda_2(a) < \dots \rightarrow \infty$. Further, each $\lambda_n(a)$ is a continuous function of a on any interval $0 < a_1 \leq a \leq a_2$. Our problem, then, is to study those values of a for which at least one member of the sequence of eigenvalues has the form $\lambda_n(a) = m^2 - 1$. Now let $R = \max r(x)$ and rewrite problem (1),

$$(2) \quad \left(\frac{y'}{p} \right)' + \omega \frac{q}{(1 - (R - r)/(a + R))p^2} y = 0, \quad y(0) = y(b) = 0,$$

where $\omega = \lambda/(a + R)$. The n th eigenvalue of problem (2) is therefore given by the formula $\omega_n(a) = \lambda_n(a)/(a + R)$. Now the coefficient of ωy is a decreasing function of a . Hence, if $a_1 < a_2$, a known theorem [1, p. 357] tells us that $\omega_n(a_1) \leq \omega_n(a_2)$, i.e.,

$$(3) \quad \frac{\lambda_n(a_1)}{a_1 + R} \leq \frac{\lambda_n(a_2)}{a_2 + R}.$$

Presented to the Society, September 3, 1953; received by the editors June 10, 1953 and, in revised form, July 16, 1953.

Finally, let us note that, for each a , $\lim_n \lambda_n(a)n^{-2}$ exists and is positive [1, p. 361]. It therefore follows that

$$(4) \quad \lim_{n \rightarrow \infty} \frac{\lambda_n(a)}{\lambda_{n+1}(a)} = 1.$$

We are now in a position to prove our main result.

THEOREM. *Let a_1 and a_2 be any two given numbers such that $0 < a_1 < a_2$. Then there exists an integer $m > 1$ and a number a , where $a_1 \leq a \leq a_2$, such that $\lambda_n(a) = m^2 - 1$ for some n .*

PROOF. Since $a_1 < a_2$, we have $(a_1 + R)/(a_2 + R) < 1$. As a result of (4) there exists an integer N such that for all $n \geq N$

$$(5) \quad \frac{\lambda_n(a_1)}{\lambda_{n+1}(a_1)} > \frac{(a_1 + R)}{(a_2 + R)}.$$

Now pick m so that $m^2 - 1 \geq \lambda_N(a_1)$ and let $n (\geq N)$ be the number such that $\lambda_n(a_1) \leq m^2 - 1 < \lambda_{n+1}(a_1)$. From the inequalities (3) and (5) we can conclude that $\lambda_{n+1}(a_1) \leq \lambda_n(a_2)$ so that $\lambda_n(a_1) \leq m^2 - 1 < \lambda_n(a_2)$. But $\lambda_n(a)$ is a continuous function of a on the interval $[a_1, a_2]$, so there must be a value of a in that interval for which $\lambda_n(a) = m^2 - 1$.

BIBLIOGRAPHY

1. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. 1, 2d ed., Berlin, 1931.
2. T. Minagawa and T. Rado, *On the infinitesimal rigidity of surfaces of revolution*, Math. Zeit. vol. 59 (1953) pp. 151-163.
3. E. Rembs, *Über die Verbiegung parabolisch berandeter Flächen negativer Krümmung*, Math. Zeit. vol. 35 (1932) pp. 529-535.

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