TWO THEOREMS ON ALMOST SURE CONVERGENCE¹

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1. Introduction. In [1]³ the author used two theorems which are presented for their fairly general interest in probability theory. Both are closely related to the theory of martingales, in fact the second is a straightforward generalization of the semi-martingale convergence theorem due to Doob [2]. For a detailed discussion of martingales see [2].

2. A convergence theorem. In this section we use a lemma due to Loève [3]. We state it here as

LEMMA 1. *Let $\{V_n\}$ be a sequence of random variables such that $\sum_{n=1}^{\infty} E\{V_n^2\} < \infty$. Then $\sum_{j=1}^n [V_j - E\{V_j | V_1, \dots, V_{j-1}\}]$ converges a.s. (almost surely).*

Let $\{X_n\}$ be a sequence of random variables and define the sequence $\{Y_n\}$ by $Y_1 = X_1$, $Y_n = X_n - X_{n-1}$ for $n > 1$. Denote the variance of Y_n by σ_n^2 . Then we have

LEMMA 2. *If $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$, then $X_n - \sum_{j=1}^n E\{X_j - X_{j-1} | X_1, \dots, X_{j-1}\}$ converges a.s.*

PROOF. Let $V_j = Y_j - E\{Y_j | Y_1, \dots, Y_{j-1}\}$. Then $E\{V_j^2\} \leq \sigma_j^2$ and Lemma 1 applies. But $E\{V_j | V_1, \dots, V_{j-1}\} = 0$ a.s., since the σ -field determined by $V_1, \dots, V_{j-1}$ coincides with the σ -field determined by $X_1, \dots, X_{j-1}$. Hence $\sum_{j=1}^n V_j$ converges a.s. and this is clearly equivalent with the conclusion of the lemma.

THEOREM 1. *Let $\{X_n\}$ be a sequence of random variables such that*

$$(2.1) \quad \sum_{n=1}^{\infty} \sigma_n^2 < \infty.$$

(2.2) *For every $\epsilon > 0$ there exists a nonnegative integer valued random variable $N(\epsilon)$ such that $n \geq N(\epsilon)$ and $|X_n| \geq \epsilon$ implies $X_n E\{Y_{n+1} | Y_1, \dots, Y_n\} \leq 0$ a.s.*

Then a necessary and sufficient condition that X_n converge a.s. is

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³ Numbers in brackets refer to the bibliography at the end of the paper.

that $E\{Y_n | Y_1, \dots, Y_{n-1}\}$ converge a.s. to 0.

PROOF. We have $Y_n = (X_n - \sum_{j=1}^n E\{Y_j | Y_1, \dots, Y_{j-1}\}) - (X_{n-1} - \sum_{j=1}^{n-1} E\{Y_j | Y_1, \dots, Y_{j-1}\}) + E\{Y_n | Y_1, \dots, Y_{n-1}\}$. Hence by applying (2.1) and Lemma 2 we have $E\{Y_n | Y_1, \dots, Y_{n-1}\}$ converges to 0 a.s. if and only if Y_n does. This shows necessity. Now suppose (2.1) and (2.2) hold and $E\{Y_n | Y_1, \dots, Y_{n-1}\}$ converges to 0 a.s. Then so does Y_n . Next we show

$$P\{\lim_{n \rightarrow \infty} X_n = +\infty\} = P\{\lim_{n \rightarrow \infty} X_n = -\infty\} = 0.$$

For suppose $\{X_n\}$ is a sample sequence for which the conclusion of Lemma 2 and (2.2) holds. Now if $\lim_{n \rightarrow \infty} X_n = \infty$, then for sufficiently large n , $E\{Y_{n+1} | Y_1, \dots, Y_n\} \leq 0$. But then we would have $\lim_{n \rightarrow \infty} [X_n - \sum_{j=1}^n E\{Y_j | Y_1, \dots, Y_{j-1}\}] = \infty$ which is impossible by Lemma 2. Similarly if $\lim_{n \rightarrow \infty} X_n = -\infty$. Now suppose $P\{\lim_{n \rightarrow \infty} X_n \text{ exists finite}\} < 1$. Then there exists a set of positive probability such that for every sample sequence $\{X_n\}$ in the set we have:

- (i) (2.2) holds.
- (ii) $X_n - \sum_{j=1}^n E\{Y_j | Y_1, \dots, Y_{j-1}\}$ converges to a finite number.
- (iii) $\liminf_{n \rightarrow \infty} X_n < \limsup_{n \rightarrow \infty} X_n$.
- (iv) Y_n converges to 0.

Let $\{X_n\}$ be such a sequence and assume $\limsup_{n \rightarrow \infty} X_n > 0$. (A similar argument applies when $\limsup_{n \rightarrow \infty} X_n \leq 0$.) Choose numbers a and b with

$$0 < a, \quad \liminf_{n \rightarrow \infty} X_n < a < b < \limsup_{n \rightarrow \infty} X_n.$$

Now choose N so large that the following conditions are satisfied:

- (i) $N > N(a/2)$ where $N(a/2)$ satisfies (2.2);
- (ii) $|Y_n| < a/2$ for $N < n$;
- (iii) $|X_m - X_n - \sum_{j=n+1}^m E\{Y_j | Y_1, \dots, Y_{j-1}\}| < b - a$ for $N \leq n < m$;

and choose m and n with $N \leq n < m$ such that:

- (i) $X_n < a$, $X_m > b$;
- (ii) $a \leq X_j \leq b$ for $n < j < m$.

Now $X_m - X_n > b - a$ and $X_n > a/2$. But then $\sum_{j=n+1}^m E\{Y_j | Y_1, \dots, Y_{j-1}\} \leq 0$ which is clearly a contradiction and thus proves the theorem.

3. Extension of a martingale theorem. In [2] Doob has proved the following theorem. If $\{X_n\}$ is a sequence of random variables with:

- (i) $\sup_n E\{|X_n|\} < \infty$;
- (ii) $E\{X_{n+1} - X_n | X_1, \dots, X_n\} \leq 0$ a.s.,

then X_n converges a.s. The following generalization is an immediate consequence of the theorem quoted.

THEOREM 2. *Let $\{X_n\}$ be a sequence of random variables satisfying*

$$(3.1) \quad \sup_n E\{|X_n|\} < \infty,$$

$$(3.2) \quad \sum_{n=1}^{\infty} E\{[E\{X_{n+1} - X_n | X_1, \dots, X_n\}]^+\} < \infty.$$

Then X_n converges a.s.

As usual we define X^+ by $X^+ = [X + |X|]/2$. One interesting aspect of Theorem 2 is that (3.2) is in a sense the best condition of its kind. To see this let $\{q_n\}$, $n=1, 2, \dots$, be a sequence of numbers with $0 \leq q_n \leq 1$. We construct a nonstationary Markov chain $\{X_n\}$ by letting X_0 assume the values ± 1 with probability $1/2$ respectively and letting X_n assume the values ± 1 with the transition probabilities

$$\begin{aligned} P\{X_n = 1 | X_{n-1} = 1\} &= p_n, & P\{X_n = -1 | X_{n-1} = 1\} &= q_n, \\ P\{X_n = 1 | X_{n-1} = -1\} &= q_n, & P\{X_n = -1 | X_{n-1} = -1\} &= p_n, \end{aligned}$$

where $p_n = 1 - q_n$. Now define the sequence of events $\{A_n\}$ by $A_n = \{(X_{n-1} = 1, X_n = -1) \text{ or } (X_{n-1} = -1, X_n = 1)\}$. Now it is clear that X_n converges a.s. if and only if $P\{A_n \text{ occurs infinitely often}\} = 0$. By direct computation it is easily verified that the A_n are independent events and that $P\{A_n\} = q_n$. Hence it follows from the Borel-Cantelli lemma that X_n converges a.s. if and only if $\sum_{n=1}^{\infty} q_n < \infty$. Furthermore, it is easily seen that $E\{[E\{X_{n+1} - X_n | X_1, \dots, X_n\}]^+\} = q_{n+1}$. Hence we find that for the family of Markov chains thus defined (3.2) is both necessary and sufficient, so that no weakening of (3.2) involving the summands is possible.

REFERENCES

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