A NOTE ON THE WAVE EQUATION

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It is well known that if u is a regular harmonic function and if E_1 and E_2 are confocal solid ellipsoids contained in the region where u is defined, then the mean value of u over E_1 is equal to the mean value of u over E_2 .¹ In this note we shall give a generalization of this result for solutions $u(x, t) = u(x_1, \dots, x_n, t)$ of the wave equation

$$\sum_{i=1}^n u_{x_i x_i} = u_{tt}.$$

Let P and Q be two points in (xt) space, and let Γ_P and Γ_Q be the corresponding characteristic cones. We suppose that

$$k(P, Q) \equiv (t_P - t_Q)^2 - \sum (x_P - x_Q)^2 > 0$$

where (x_P, t_P) and (x_Q, t_Q) are the coordinates of P and Q respectively. It follows that the intersecting nappes of Γ_P and Γ_Q bound a finite region R . We shall say that an ellipsoid E is inscribed in R if E is contained in R and if for any two points P', Q' such that $k(P', Q') < k(P, Q)$, the corresponding region R' does not contain E .

Now note that there exists a Lorentz transformation² $\mathcal{L}$ making the line PQ a time axis, and finally let L denote the plane of intersection of Γ_P and Γ_Q . Then we have the following theorem.

If E_1 and E_2 are solid ellipsoids inscribed in R , then the mean value of $u(x, t)$ over E_1 is equal to the mean value of $u(x, t)$ over E_2 , provided that the cross sections of E_1 and E_2 by the plane L are transformed into confocal ellipsoids by $\mathcal{L}$.³

Since the Lorentz transformation $\mathcal{L}$ clearly leaves the "inscribed" relation invariant, it follows that the theorem may be reduced to the special case when PQ is a time axis, and $\mathcal{L}$ is the identity. In this case one readily verifies that PQ is also a principal axis of E_1 and E_2 .

By taking a suitable coordinate system we may therefore assume in our proof that the principal axes of E_1 and E_2 coincide with the coordinate axes, and take P and Q to be $(0, \dots, 0, \tau)$ and

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¹ L. Asgeirsson, Math. Ann. vol. 113 (1936) pp. 321-346, or Courant-Hilbert, *Methoden der mathematischen Physik*, vol. II, pp. 417-424.

² I.e. a linear transformation of (xt) leaving the form $t^2 - \sum x_i^2$, and hence the wave equation invariant.

³ In other words the cross sections of E_1 and E_2 by L are confocal ellipsoids in the metric $d(P, Q) = (-k(P, Q))^{1/2}$.

$(0, \dots, 0, -\tau)$ respectively. Then L becomes the plane $t=0$, and E_1 and E_2 may be represented by

$$\sum_{i=1}^n \frac{x_i^2}{a_i^2} + \frac{t^2}{b^2 - 1} \leq r^2 \quad \text{and} \quad \sum_{i=1}^n \frac{x_i^2}{a_i^2 - 1} + \frac{t^2}{b^2} \leq r^2,$$

where

$$a_1 \geq a_2 \geq \dots \geq a_n > 1, \quad b > 1, \quad \text{and} \quad r = \tau(a_1^2 + b^2 - 1)^{-1/2}.$$

The last relations guarantee that both ellipsoids are inscribed in R .

Now consider the function

$$v(\xi, \eta) = v(\xi_1, \dots, \xi_{n+1}, \eta_1, \dots, \eta_{n+1}) = u(x_1, \dots, x_n, t),$$

where

$$\begin{aligned} x_i &= a_i \xi_i + (a_i^2 - 1)^{1/2} \eta_i, & i &= 1, \dots, n, \\ t &= b \eta_{n+1} + (b^2 - 1)^{1/2} \xi_{n+1}. \end{aligned}$$

Then

$$\sum_{i=1}^{n+1} v_{\xi_i \xi_i} = \sum_{i=1}^{n+1} v_{\eta_i \eta_i}.$$

From the Asgerisson mean value theorem,¹ we obtain

$$\begin{aligned} \int \dots \int_{\sum_{i=1}^{n+1} \xi_i^2 \leq r^2} v(\xi, 0) d\xi_1 \dots d\xi_{n+1} \\ = \int \dots \int_{\sum_{i=1}^{n+1} \eta_i^2 \leq r^2} v(0, \eta) d\eta_1 \dots d\eta_{n+1}, \end{aligned}$$

or, expressed in terms of the (x, t) variables,

$$\frac{\int \dots \int_{E_1} u(x, t) dx_1 \dots dx_n dt}{\int \dots \int_{E_1} dx_1 \dots dx_n dt} = \frac{\int \dots \int_{E_2} u(x, t) dx_1 \dots dx_n dt}{\int \dots \int_{E_2} dx_1 \dots dx_n dt} \quad \text{Q.E.D.}$$

One may of course find analogous results for equations

$$\sum_{i=1}^n u_{x_i x_i} = \sum_{i=1}^m u_{y_i y_i},$$

but the interpretation is no longer so apparent.