ON ORDERED DIVISION RINGS¹

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In this paper the relation between ordered fields and fields with valuations that was established by Baer [1] and Krull [4] is generalized to ordered division rings. Also some extension theory for ordered division rings is obtained by considering the induced group extensions.

1. Ordered groups. Suppose Γ is a simply ordered set and for each γ in Γ let R^γ be an additive group. Consider the set $H = H(\Gamma, R^\gamma)$ of all vectors $(\dots, r^\gamma, \dots)$ for which r^γ is in R^γ and *almost every* r^γ is zero (i.e., the set of γ 's for which $r^\gamma \neq 0$ is inversely well ordered). H is a group w.r.t. (with respect to) vector addition, called the Γ -sum of the R^γ . If the R^γ are ordered groups (notation o -group), then H is an o -group w.r.t. the following definition of order: h in H is positive if $h \neq 0$ and the nonzero component with greatest γ is positive.

For the rest of this section let G denote an additive o -group. The set Γ_G of all pairs of convex subgroups G^γ, G_γ of G where G^γ covers G_γ is simply ordered by inclusion, and each G^γ/G_γ is o -isomorphic to a subgroup of the additive group of real numbers. Thus G determines a Γ_G -sum $H = H(\Gamma_G, G^\gamma/G_\gamma)$, and H is an abelian o -group. The *value* $V(g)$ of $g \neq 0$ in G is the γ in Γ_G for which g is in G^γ but not in G_γ . If a and b are elements of H for which $b^\gamma = a^\gamma$ for all $\gamma > \beta$, $b^\gamma = 0$ for all $\gamma \leq \beta$, and $a^\beta \neq 0$, then b is called the β th head of a .

LEMMA 1.1. *There exists a 1-1 order and value-preserving mapping π of the set G into the set H such that for every g in G :*

- (a) *If $V(g) = \gamma$, then $(g\pi)^\gamma = G_\gamma + g$. $0\pi = 0$.*
- (b) *If a is the β th head of $g\pi$, then there exists an h in G for which $h\pi = a$, $V(g - h) = \beta$, and $G_\beta + g - h = (g\pi)^\beta$.*

PROOF. There exists a maximal subset K of G for which the lemma holds (apply Zorn's lemma). Suppose that d is in G but not in K . To complete the proof it is sufficient to show that π can be extended to $\{K, d\}$. Let $V(d) = \delta$. Consider the set Δ of all elements x in H with $V(x) = \delta$, $x^\delta = G_\delta + d$, and such that for every β th head y of x there exists an element $h(\beta)$ in K for which $h(\beta)\pi = y$, $V(d - h(\beta)) = \beta$,

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and $G_\beta + d - h(\beta) = x^\beta$. Δ is not void since it contains the element $(0, \dots, 0, G_\delta + d, 0, \dots, 0)$. We define a partial order over Δ as follows: $x < y$ if x is a head of y . It follows by an application of Zorn's lemma that there exists a maximal element q of Δ . Assume (by way of contradiction) that q is in $K\pi$, and let k be the unique element in K for which $k\pi = q$. Denote the totality of γ in Γ_G for which $q^\gamma \neq 0$ by $\Gamma(q)$. For each γ in $\Gamma(q)$ there exists an $h(\gamma)$ in K for which $h(\gamma)\pi$ = the γ th head of q . Moreover $V(d - h(\gamma)) = V(k - h(\gamma)) = V(h(\gamma) - k) = \gamma$, $G_\gamma + d - h(\gamma) = q^\gamma$, and $G_\gamma + h(\gamma) - k = -q^\gamma$. Thus $\alpha = V(d - k) = V(d - h(\gamma) + h(\gamma) + k) < \gamma$ for all γ in $\Gamma(q)$. Let s be the element in H such that $s^\gamma = q^\gamma$ for all $\gamma \neq \alpha$, and $s^\alpha = G_\alpha + d - k$. Clearly $q < s$, and s belongs to Δ , a contradiction. Therefore q is not in $K\pi$. Define $d\pi = q$. π is a 1-1 value-preserving mapping of $\{K, d\}$ into H that satisfies (a) and (b). To complete the proof we must show that π preserves order.

Suppose that $d < h$ for h in K . If $h = 0$ or $V(d) < V(h)$ or $V(d) = V(h)$ and $G_\delta + d < G_\delta + h$, then $d\pi < h\pi$, since π satisfies (a). Assume that $d \equiv h \pmod{G_\delta}$; then there exists a β in Γ_G such that $(d\pi)^\beta \neq (h\pi)^\beta$ and $(d\pi)^\gamma = (h\pi)^\gamma$ for all $\gamma > \beta$. Let k be the element in K such that $k\pi = \beta$ th head of $h\pi$ (and of $d\pi$). Then $G_\beta + h - k = (h\pi)^\beta$ and $G_\beta + d - k = (d\pi)^\beta$. Thus if $(d\pi)^\beta > (h\pi)^\beta$, then $G_\beta + d - k > G_\beta + h - k$, but this means that $d > h$, a contradiction. Therefore $d\pi < h\pi$. By a similar argument it follows that $d > h$ implies $d\pi > h\pi$.

Suppose that S is a subgroup of G . G is an *a-extension* of S if for every $g > 0$ in G there exist an $s > 0$ in S and a positive integer n such that $ns \leq g \leq (n+1)s$. G is a *c-extension* of S if for every γ in Γ_G , $G^\gamma = G_\gamma + (S \cap G^\gamma)$, G is *a-closed* (*c-closed*) if it does not admit any proper *a*-extensions (*c*-extensions). It is clear that G is an *a-extension* of S if and only if $\Gamma_G = \Gamma_S$ (i.e., $(S \cap G^\gamma)/(S \cap G_\gamma) \neq 0$ for every γ in Γ_G). Moreover a *c-extension* is an *a-extension*, hence *a-closure* implies *c-closure*.

COROLLARY I. *An upper bound for the number of elements in any a-extension of G is C^Δ where C and Δ are the cardinal numbers of the set of real numbers and the set Γ_G respectively.*

PROOF. If F is an *a-extension* of G , then there exists a 1-1 correspondence of F into $H(\Gamma_G, F^\gamma/F_\gamma)$. But there exists an isomorphism of $H(\Gamma_G, F^\gamma/F_\gamma)$ into $H(\Gamma_G, R)$ where R = reals, and the number of elements in $H(\Gamma_G, R) \leq C^\Delta$. By the usual transfinite arguments we have

COROLLARY II. *There exists an a-closed a-extension and a c-closed c-extension of G .*

If G is c -closed and every G^γ/G_γ is o -isomorphic to R , then G is a -closed. For suppose that F is an a -extension of G ; then $\Gamma_G = \Gamma_F$ and the o -isomorphism $\pi(\gamma)$ that maps $G_\gamma + a$ of G^γ/G_γ upon $F_\gamma + a$ of F^γ/F_γ is onto (otherwise we would have an o -isomorphism of the additive group of real numbers into a proper subgroup). Thus F is a c -extension of G , hence $F = G$. If G is abelian, then any two a -closed a -extensions are equivalent, and o -isomorphic to $H(\Gamma_G, R)$. For a proof of this and other results on abelian o -groups see Conrad [2].

2. The natural valuation of an ordered division ring. The additive group of a division ring D will be denoted by ${}^+D$, and the multiplicative group by D^* . D can be ordered if and only if one of the following conditions is satisfied.

(i) D^* has a subgroup of index 2 which is closed w.r.t. addition.

(ii) -1 cannot be represented as a sum of elements of the form $a_1^2 a_2^2 \cdots a_n^2$, where a_i is in D for $i = 1, 2, \dots, n$ (Szele [7]).

For the rest of this paper let D denote an ordered division ring. Consider any two elements α, β in $\Gamma_D = \Gamma_{+D}$, and elements a, b in D such that $V(a) = \alpha$ and $V(b) = \beta$. Define $\alpha + \beta = \gamma$ where $\gamma = V(ab)$. It is easy to prove that Γ_D is an o -group (w.r.t. the previously defined order in the set Γ_D). Moreover V is a valuation of D , called the *natural valuation of D* .² The identity $V(1)$ of Γ_D will be denoted by θ . Finally we define $V(0) = -\infty$ so that $V(0) < \gamma$, and $\gamma - \infty = -\infty + \gamma = -\infty - \infty = -\infty$ for every γ in Γ_D .

Suppose that R is a division ring with valuation f and value group Γ . For every γ in Γ we define $R^\gamma = R^\gamma(f) = \{r \text{ in } R \mid f(r) \leq \gamma\}$, and $R_\gamma = R_\gamma(f) = \{r \text{ in } R \mid f(r) < \gamma\}$. Clearly R^γ and R_γ are abelian groups w.r.t. addition. R^θ is the *valuation ring of f in R* and R^θ/R_θ is the *residue class ring of R , w.r.t. f* .

LEMMA 2.1. *For every a, b, c in R and γ in Γ we have: (a) If $f(a) = f(b) = \gamma$, then $a = cb$ where $f(c) = \theta$, and $a \equiv b \pmod{R_\gamma}$ if and only if $c \equiv 1 \pmod{R_\theta}$. (b) There exists a group isomorphism $\pi(\gamma)$ of R^θ/R_θ onto R^γ/R_γ . (c) The mapping of h in $H(\Gamma, R^\theta/R_\theta)$ upon $h\pi$ in $H(\Gamma, R^\gamma/R_\gamma)$ where $(h\pi)^\gamma = h^\gamma\pi(\gamma)$ is an isomorphism.*

The proof of (a) follows directly from the definition of a valuation, and (c) is an immediate consequence of (b). Suppose that x is an element of R for which $f(x) = \gamma$. Every element in R^θ/R_θ is of the form $R_\theta + c$ where $f(c) \leq \theta$. Define the mapping $\pi = \pi(x, \gamma): (R_\theta + c)\pi = R_\gamma$

² Here $V(a+b) \leq \text{Max } [V(a), V(b)]$ instead of the usual condition $V(a+b) \geq \text{Min } [V(a), V(b)]$ (Schilling [6, p. 9]). We prefer this because it makes use of the natural order of Γ_D .

$+cx$. It can easily be shown that π is an isomorphism of R^θ/R_θ onto R^γ/R_γ . If $R=D$ and $f=V$, then by choosing x positive we have

COROLLARY. *There exists an o-isomorphism of D^θ/D_θ onto D^γ/D_γ for every γ in Γ_D . The standard arguments show that D^θ/D_θ is o-isomorphic to one and only one subfield of the real numbers which we shall denote by R_D .*

THEOREM 2.2. *A division ring R can be ordered if and only if (1) there exists a valuation f of R for which $R^\theta(f)/R_\theta(f)$ can be ordered so that every element of the form $R_\theta(f) + a_1^2 \cdots a_n^2$ is positive, where a_i is in R for $i=1, 2, \dots, n$ and $f(a_1^2 \cdots a_n^2) = \theta$. Moreover R can be ordered so that f is the natural valuation if and only if (2) the ordering of $R^\theta(f)/R_\theta(f)$ is Archimedean.*

PROOF. If R is ordered, then the natural valuation satisfies conditions (1) and (2). Conversely suppose that (1) is satisfied, and let Γ be the value group of f . Let S be the set of all elements in R^* of the form $d_0 d_1^2 \cdots d_n^2$, where d_i is in R^* for $i=0, 1, \dots, n$, $f(d_0) = \theta$, and $R_\theta + d_0$ is positive (once again let R^γ, R_γ denote $R^\gamma(f), R_\gamma(f)$). By (a) of Lemma 2.1 an element x of R^* for which $f(x) = \gamma$ is in S if and only if $x \equiv d_0 d_1^2 \cdots d_n^2 \pmod{R_\gamma}$ (same conditions on the d_i). If the element $R_\theta + s$ in R^θ/R_θ is positive, and g is any element in R^* , then $R_\theta + g^{-1}sg$ is positive. Otherwise $R_\theta + g^{-1}sg = R_\theta + (g^{-1}sg)^2(g^{-1})^2$ is negative, but this contradicts condition (1). It follows immediately that S is a normal subgroup of R^* . S is closed w.r.t. addition. For suppose $d = d_0 d_1^2 \cdots d_n^2$ and $c = c_0 c_1^2 \cdots c_m^2$ are elements of S , and $f(d) = \delta \leq \gamma = f(c)$. If $\delta < \gamma$, then $d + c \equiv c \pmod{R_\gamma}$, and $f(d + c) = \gamma$. If $\delta = \gamma$, then $d_0 d_1^2 \cdots d_n^2 = g c_0 c_1^2 \cdots c_m^2$, where $f(g) = \theta$. Hence $d_1^2 \cdots d_n^2 (c_m^{-1})^2 \cdots (c_1^{-1})^2 = d_0^{-1} g c_0$. Thus by (1) $R_\theta + d_0^{-1} g c_0$ is positive, hence $R_\theta + (g + 1)c_0$ is positive. But $d + c = (g + 1)c_0 c_1^2 \cdots c_m^2$. Hence in either case $d + c$ is in S . -1 is not in S . Otherwise $-1 = d_0 d_1^2 \cdots d_n^2$, hence $R_\theta - d_0^{-1} = R_\theta + d_1^2 \cdots d_n^2$ is negative, but this contradicts (1).

By Zorn's lemma there exists a maximal normal subgroup P of R^* for which $S \leq P$, -1 is not in P , and P is closed w.r.t. addition. Szele [7] proved that P is a subgroup of index 2. Hence R can be ordered (let P be the set of positive elements), and this ordering extends the order of R^θ/R_θ . Now suppose that condition (2) is satisfied. To prove that f is the natural valuation it is sufficient to show that $R^\theta(f) = R^\theta(V)$.³ Thus it is sufficient to show that for every $x > 0$ in R , $f(x) \leq \theta$ if and only if $V(x) \leq \theta$. If $f(x) < \theta$, then $1 - mx \equiv 1 \pmod{R_\theta}$ for

³ This is a consequence of Lemma 6, p. 10 in Schilling [6]. The author wishes to thank the referee for indicating this method of proof.

all integers m , so that $1 - mx$ is in $S \leq P$. Thus $mx < 1$ for all m , hence $V(x) < \theta$. If $f(x) = \theta$, then since R^θ/R_θ is Archimedean there exist positive integers m and n such that $mx + R_\theta > 1 + R_\theta$ and $n1 + R_\theta > x + R_\theta$. Thus $mx > 1$ and $n1 > x$, hence $V(x) = \theta$. Finally if $f(x) > \theta$, then $f(x^{-1}) < \theta$, hence $V(x^{-1}) < \theta$ and $V(x) > \theta$.

3. Extension theorems.

LEMMA 3.1. *Suppose that R is a division ring with valuation f and value group Γ . Then there exists an isomorphism π of ${}^+R$ into the Γ -sum $H(\Gamma, R^\gamma(f)/R_\gamma(f))$ [which is isomorphic to $H(\Gamma, R^\theta(f)/R_\theta(f))$]. Moreover if $f(a) = \alpha$ for a in R^* , then $(a\pi)^\alpha = R_\alpha + a$ and $(a\pi)^\gamma = R_\gamma$ for all $\gamma > \alpha$.*

PROOF. Let P be the prime field of R . Then ${}^+R$, $R^\gamma(f)$, and $R_\gamma(f)$ are abelian operator groups w.r.t. P . The pairs $R^\gamma(f)$, $R_\gamma(f)$ satisfy the definition of a Γ -group (Conrad [2]). Hence this lemma is a special case of the main embedding theorem.

COROLLARY. *If $R = D$ and $f = V$, then π preserves order and ${}^+D\pi$ is a c -subgroup of H .*

Suppose that S and R are division rings with valuations g and f , and value groups Γ_g and Γ_f respectively. Clearly S is an immediate extension of R (Schilling [6, p. 36]) if and only if $S \geq R$, $\Gamma_g = \Gamma_f$, and $S^\gamma(g) = S_\gamma(g) + R^\gamma(f)$ for every γ in Γ_g .

THEOREM 3.2. *For each division ring R with valuation f there exists a maximal immediate extension.*

PROOF. If S is an immediate extension of R , then ${}^+S$ is isomorphic to a subgroup of the Γ -sum $H(\Gamma_f, R^\theta(f)/R_\theta(f))$. A transfinite argument completes the proof.

An ordered division ring E is an a -extension (m -extension) of D if $E \geq D$ and ${}^+E$ is an a -extension of ${}^+D$ (Γ_E is an a -extension of Γ_D).

THEOREM 3.3. *D is a -closed if and only if R_D is the field of reals and D is maximally complete.*

PROOF. Assume that D is a -closed, then D is maximally complete since an immediate extension is an a -extension. Neumann [5] proved that each ordered division ring G admits an a -extension $\bar{G}$ that contains a subfield o -isomorphic to the reals. Clearly $R_{\bar{G}} = \text{reals}$, hence $R_D = \text{reals}$. Conversely assume that D is maximally complete and $R_D = \text{reals}$. Then by the corollary to Lemma 3.1, for any a -extension E of D , ${}^+E$ is isomorphic to a c -subgroup of the Γ -sum $H(\Gamma_D, R_E)$

$= H(\Gamma_D, R_D)$. Hence any a -extension of D is an immediate extension. Therefore D is a -closed.

COROLLARY. *There exists an a -closed a -extension of D .*

THEOREM 3.4. *There exists an m -closed m -extension of D .*

PROOF. If E is an m -extension of D , then ${}^+E$ is isomorphic to a subgroup of $H(\Gamma_E, R)$ where R =reals and Γ_E is an a -extension of Γ_D . By Corollary I of Lemma 1.1 the number of elements in any such Γ_E is bounded and the bound is determined by D . A transfinite argument completes the proof.

REMARKS ON o -FIELDS. The Γ -sum $H = H(\Gamma_D, R_D)$ is the additive group of the ordered division ring $\overline{H}$ of all formal power series over Γ_D with coefficients in R_D . $\overline{H}$ is maximally complete. $\overline{H}$ is a -closed if and only if R_D =reals. $\overline{H}$ is m -closed if and only if Γ_D is a -closed and R_D =reals. Using Theorem 6, p. 226, Schilling [6], the following well known results can be proved. Each o -field F is o -isomorphic to a subfield of the field $\overline{H}(\Gamma_F, R)$ where R =reals. $\overline{H}$ is the a -closure of the image of F under this isomorphism. Also F is isomorphic to a subfield of the field $\overline{H}(\Gamma, R)$ where Γ is the a -closure of Γ_F . $\overline{H}$ is the m -closure of the image of F under this isomorphism. Finally, F is m -closed if and only if Γ_F and F are a -closed.

BIBLIOGRAPHY

1. R. Baer, *Über nichtarchimedisch geordnete Körper*, Sitzungsberichte der Heidelberger Akademie der Wissenschaften vol. 8 (1927).
2. P. Conrad, *Embedding theorems for abelian groups with valuations*, Amer. J. Math. vol. 75 (1953) pp. 1-29.
3. I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. vol. 9 (1942) pp. 303-321.
4. W. Krull, *Allgemeine Bewertungstheorie*, Journal für Mathematik vol. 167 (1932) pp. 160-196.
5. B. H. Neumann, *On ordered division rings*, Trans. Amer. Math. Soc. vol. 66 (1949) pp. 202-252.
6. O. F. G. Schilling, *The theory of valuations*, American Mathematical Society, 1950.
7. T. Szele, *On ordered skew fields*, Proc. Amer. Math. Soc. vol. 3 (1952) pp. 410-413.

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