BIBLIOGRAPHY

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A REPRESENTATION PROBLEM¹

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1. Introduction and summary. The problem solved in this paper is the following. For a fixed number a , $0 < a < 1$, what functions $f(x)$ on $0 \leq x \leq 1$ have a representation

$$(1) \quad f(x) = \sum_1^{\infty} c_n \phi_n(x),$$

where $c_n \geq 0$, $\sum_1^{\infty} c_n$ converges, and each ϕ_n is the characteristic function of a subset of $0 \leq x \leq 1$ of Lebesgue measure a ? Clearly any f satisfying (1) satisfies

$$(2) \quad 0 \leq f(x) \leq \frac{1}{a} \int_0^1 f(x) dx \quad \text{for all } x,$$

since $f(x) \leq \sum_1^{\infty} c_n = (1/a) \int_0^1 f(x) dx$. The result of this paper is that (2) is sufficient as well as necessary for a function f to admit a representation (1).

The problem arises in connection with a submarine search game discussed by Morse and Kimball [1], which runs as follows. A submarine must pass through a channel of length 1, and must choose a part of the channel of length a at which to be on the surface. An enemy plane must choose a point x along the channel at which to look for the submarine. The submarine escapes if it is submerged at the x chosen by the plane; otherwise it has a probability $h(x)$ of being detected. Thus a pure strategy for the submarine is a subset S of $0 \leq x \leq 1$ of Lebesgue measure a , and its detection probability for the given S as a function of x is $h(x)\phi(x)$, where ϕ is the characteristic

Received by the editors August 3, 1953.

¹ This paper was written under contract Nonr-1197(00) with the Office of Naval Research.

function of S . If the submarine uses a mixed strategy, mixing $S_1, S_2, \dots$ with probabilities $c_1, c_2, \dots$, its detection probability function is

$$\sum_1^{\infty} c_n h(x) \phi_n(x) = h(x) f(x),$$

where ϕ_n is the characteristic function of S_n and $f(x)$ is given by (1). Morse and Kimball [1] solve the game by supposing that the submarine can choose any $f(x)$ with $0 \leq f \leq 1$ and $\int_0^1 f(x) dx = a$; our result justifies their procedure, since it shows that precisely these functions f admit a representation (1) with $\sum_1^{\infty} c_n = 1$.

Our result incidentally shows that only countable mixtures need be used; for instance the class of f 's representable by

$$(3) \quad f(x) = \int_0^1 \phi(x, y) dm(y),$$

where $\phi(x, y)$ is a Borel-measurable function of x, y which for each y is the characteristic function of an x -set of Lebesgue measure a and m is a finite measure defined for Borel sets of the unit square, is also described by (2), since (3) reduces to (1) for discrete measures and every f satisfying (3) must satisfy (2).

2. Proof of the theorem. We restate our result in more general form, as follows.

THEOREM. *Let m be any nonatomic finite measure on a Borel field $\mathcal{B}$ of subsets of a space X and let a be any number with $0 < a < m(X)$. The functions f admitting a representation*

$$(4) \quad f = \sum c_n \phi_n,$$

where $c_n \geq 0$, $\sum_1^{\infty} c_n$ converges, and each ϕ_n is the characteristic function of a set $S_n \in \mathcal{B}$ with $m(S_n) = a$, are precisely the $\mathcal{B}$ -measurable functions f satisfying

$$(5) \quad 0 \leq f(x) \leq \frac{1}{a} \int f dm \quad \text{for all } x.$$

PROOF. That (4) implies (5) is clear, as remarked in §1. For the converse, we shall, supposing (5), first obtain an almost everywhere representation (4), then modify each ϕ_n on a set of measure zero to obtain the everywhere representation.

For the almost everywhere representation, the appropriate class of functions is that satisfying

$$(5ae) \quad 0 \leq \operatorname{essmin} f \leq \operatorname{essmax} f \leq \frac{1}{a} \int f dm.$$

LEMMA. For any f satisfying (5ae), there is a set S with $m(S) = a$ such that

$$(6) \quad u = \operatorname{essmax}_{x \notin S} f \leq v = \operatorname{essmin}_{x \in S} f.$$

The numbers u and v depend only on f and not on the particular S chosen.

The proof of the lemma is straightforward; we consider the function $F(t) = m\{f \geq t\}$, which decreases from $m(X)$ to 0 as t goes from 0 to ∞ . If F assumes the value a , say at t^* , we choose $S = \{f \geq t^*\}$. If F does not assume the value a , say $F(t^* -) > a \geq F(t^* +)$. Then f assumes the value t^* on a set R of measure $F(t^* -) - F(t^* +)$. We choose $S = \{f > t^*\} \cup T$, where T is any subset of R with $m(T) = a - F(t^* +)$: such a T exists, since m is nonatomic. To show uniqueness of u and v : if there are two admissible sets S_1, S_2 with $m(S_1 \cap S_2 + S_2 \cap S_1) > 0$, then $m(S_1 \cap S_2) = m(S_2 \cap S_1) > 0$, and

$$\operatorname{essmin}_{x \in S_1} f \leq \operatorname{essmin}_{x \in S_1 \cap S_2} f \leq \operatorname{essmax}_{x \in S_2} f \leq \operatorname{essmin}_{x \in S_2} f,$$

so that $u(S_1) \leq u(S_2)$. Similarly $u(S_2) \leq u(S_1)$ and similarly for $v(S_1), v(S_2)$.

We continue with the proof of the theorem, denoting, for any f satisfying (5ae), by $S(f), u(f), v(f)$ an S and the u, v associated with f by the lemma. Let f satisfy (5ae), let ϕ_1 be the characteristic function of a set $S(f)$, and consider $g = f - \lambda \phi_1, \lambda \geq 0$. Since $\operatorname{essmax} g = \max(u(f), \operatorname{essmax} f - \lambda)$ and

$$\frac{1}{a} \int g dm = \frac{1}{a} \int f dm - \lambda,$$

g will satisfy (5ae) if and only if

$$\lambda \leq \min(v(f), w(f)) = \lambda(f)$$

where

$$w(f) = \frac{1}{a} \int f dm - u(f).$$

Define $\lambda_1 = \lambda(f), f_1 = f - \lambda_1 \phi_1$ and, inductively for $n > 1$, $\lambda_n = \lambda(f_{n-1}), \phi_n$ as the characteristic function of a set $S(f_{n-1}), f_n = f_{n-1} - \lambda_n \phi_n$. Then $f \geq f_1 \geq f_2 \geq \dots$, each f satisfies (5ae), and

$$f = \sum_1^n \lambda_i \phi_i + f_n.$$

Consequently

$$(7) \quad f = \sum_1^\infty \lambda_n \phi_n + f^*,$$

where $f^* = \lim f_n$. Moreover f^* satisfies (5ae), since the inequalities of (5ae) are preserved under monotone passage to the limit.

We now complete the proof for almost everywhere representation by showing that $f^* = c\phi$ almost everywhere, where c is a non-negative constant and ϕ is the characteristic function of a set of measure a . From the inequalities, valid for all f satisfying (5ae),

$$(8) \quad \int_{CS} f dm \leq v(f) [m(X) - a],$$

$$(9) \quad \int_{CS} f dm = \int f dm - \int_S f dm \leq \int f dm - au(f) = aw(f),$$

we obtain

$$(10) \quad \int_{CS} f dm \leq m(X)\lambda(f),$$

where $S = S(f)$. Applying (10) with $f = f_{n-1}$, summing over n , and using $f_{n-1} \geq f^*$ yields

$$(11) \quad \sum_1^\infty \int_{CS_n} f^* dm \leq m(X) \sum_1^\infty \lambda_n,$$

where $S_n = S(f_{n-1})$. From (11) we obtain

$$\int_U f^* dm = 0,$$

where $U = \limsup CS_n$. Since $m(U) \geq m(X) - a$, f^* vanishes except on some set S of measure a ; since f^* satisfies (5ae),

$$(12) \quad \int f dm = \int_S f dm \leq a \operatorname{essmax} f \leq \int f dm.$$

Consequently we have equality through (12) and f^* equals its essential maximum almost everywhere on S , i.e. $f^* = c\phi$ almost everywhere, where c is a non-negative constant and ϕ is the characteristic function of a set of measure a . This completes the proof for almost

everywhere representation of functions satisfying (5ae).

Now let f satisfy (5) and let $\sum_1^\infty c_n \phi_n$ be an almost everywhere representation of f . Write $d_{nk} = c_n 2^{-k}$, $\phi_{nk} = \phi_n$, so that $\sum_{n,k} d_{nk} \phi_{nk}$ is an almost everywhere representation of f . Any number t satisfying $0 \leq t \leq \sum_{n,k} d_{nk}$ is the sum of an appropriate subseries of $\sum d_{nk}$: if

$$\sum_1^{N-1} c_n + m c_N 2^{-k} \leq t < \sum_1^{N-1} c_n + (m+1) c_N 2^{-k}, \quad m = 0, 1, \dots, 2^k - 1,$$

defining $\phi_{nk} = 1$ for $n < N$, 0 for $n > N$, and $\phi_{nk} = 0$ or 1 according as m is odd or even yields $t = \sum_{n,k} d_{nk} \phi_{nk}$. Redefining ϕ_{nk} in this way on the exceptional set yields $f = \sum_{n,k} d_{nk} \phi_{nk}$ everywhere and completes the proof of the theorem.

3. Further problems. Submarines cannot choose arbitrary Borel sets S at which to be on the surface, but only more restricted sets, e.g. sums of intervals. This suggests alternative problems: what functions f have a representation (1) when the ϕ_n are required to be characteristic functions of sets S of measure a which are (A) intervals, (B) finite sums of intervals, each of length at least b , $0 < b < a$, (C) finite sums of intervals? Does an integral representation (3) yield anything new in these cases? These problems appear to be more difficult than the one solved in this paper.

REFERENCE

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