This formula also reduces to equation (1) of Ramaswami's paper when $k = 2$.

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CALIFORNIA INSTITUTE OF TECHNOLOGY

ON THE ASYMPTOTIC DISTRIBUTION OF CERTAIN SUMS

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1. **Introduction.** Let $((t)) = t - [t] - 1/2$, the brackets denoting the greatest integer function. It has been shown by Kac [2]¹ that the distribution of the sum $N^{-1/2} \sum_{0 \leq n < N} ((2^n t))$ is asymptotically normal with mean 0 and variance 1/4. In [1, §8], we considered the sum $\sum_{0 \leq n < N} ((2^n t - 1/2))$ and proved that it is bounded uniformly in t and N . It seemed of interest, therefore, to study the behavior of

$$(1) \quad S_N(f; t) = N^{-1/2} \sum_{n=0}^{N-1} f(2^n t)$$

for

$$f(t) = ((t - \beta)), \quad 0 \leq \beta < 1.$$

It is proved in §2 that the distribution is asymptotically normal, with mean 0 and variance

$$(2) \quad \sigma^2(\beta) = \sum_{n=1}^{\infty} 2^{-n} ((\beta - \beta_n))^2,$$

where β_n denotes the fractional part of $2^n \beta$. Thus $\beta = 1/2$ is the only case of zero variance. For this case we determine the limiting distribution of $N^{1/2} S_N(f; t)$.

In §3 we consider $\gamma_N(t; \alpha, \beta)$, the relative frequency with which $t_n = 2^n t - [2^n t]$, $0 \leq n < N$, falls in a given interval (α, β) . Here, again, the distribution is asymptotically normal, and an explicit formula

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¹ Numbers in brackets refer to the bibliography at the end of the paper.

is obtained for the variance. If the interval is $(0, \beta)$, the mean is β and the variance is τ^2/N , where

$$(3) \quad \tau^2 = \beta - \beta^2 + 2 \sum_{n=1}^{\infty} 2^{-n} \{ \min(\beta, \beta_n) - \beta\beta_n \}.$$

Thus $\tau^2=0$ only in the trivial cases $\beta=0, 1$.

In §4 we study the combined random variable $(S_N(f; t), S_N(g; t))$ and prove that it is asymptotically normal, for real functions f, g , which satisfy the following conditions:

$$(i) \quad f(t+1) = f(t),$$

$$(4) \quad (ii) \quad \int_0^1 f(t) dt = 0,$$

$$(iii) \quad \text{The Walsh-Fourier coefficients } c_k = O(k^{-a}), \quad a > 1/2.$$

$$(iv) \quad \text{The variance } \sigma^2 \text{ does not vanish, where}$$

$$(4a) \quad \sigma^2 = \int_0^1 f^2(t) dt + 2 \sum_{n=1}^{\infty} \int_0^1 f(t) f(2^n t) dt.$$

These conditions² have been shown by Kac [3] to be sufficient for the asymptotic normality of $S_N(f; t)$, with mean 0 and variance σ^2 .

In §5 we apply the result of §4 to the special cases $f(t) = ((t-\beta))$, $g(t) = ((t-\beta'))$ and

$$S_N(f; t) = N^{1/2}(\nu_N(t; \alpha, \beta) - (\beta - \alpha)),$$

$$S_N(g; t) = N^{1/2}(\nu_N(t; \alpha', \beta') - (\beta' - \alpha')),$$

obtaining explicit formulas for the correlation coefficients, the variances having been computed in §2 and §3.

2. The function $((t-\beta))$. We begin with a theorem due to Kac [3, Theorem 1]:

THEOREM 1. *If $f(t)$ satisfies conditions (4), then the measure of the set in $(0, 1)$ on which $S_N(f; t) < b$ approaches, as $N \rightarrow \infty$, the integral³*

$$(5) \quad \Phi(b; \sigma) = \frac{1}{\sigma(2\pi)^{1/2}} \int_{-\infty}^b e^{-u^2/2\sigma^2} du.$$

Now let $f(t) = ((t-\beta))$, $0 \leq \beta < 1$. Clearly f satisfies the first three

² Kac uses ordinary Fourier coefficients, but the modification is almost trivial; indeed, the proof is somewhat simplified by the use of Walsh series.

³ Kac proves somewhat more, since he includes functions satisfying Hölder's condition of order $\leq 1/2$. The finiteness of σ is guaranteed by Lemma 1 of [3].

conditions of (4), so it remains merely to compute $\sigma^2 = \sigma^2(\beta)$ from (4a). For this purpose, we define

$$(6) \quad G_\beta(t) = \begin{cases} 1, & 0 \leq t < \beta, \\ 0, & \beta \leq t < 1, \end{cases}$$

$$G_\beta(t+1) = G_\beta(t), \quad -\infty < t < \infty.$$

It is easy to see that

$$f(t) = ((t)) + G_\beta(t) - \beta.$$

To facilitate the computation, we state two lemmas:

LEMMA 1. Let $\psi_k(t)$ denote the k th Walsh function and let $J_k(u) = \int_0^u \psi_k(t) dt$. Then

$$(a) \quad ((t)) = -\frac{1}{4} \sum_{k=0}^{\infty} 2^{-k} \psi_{2^k}(t),$$

$$(b) \quad G_\beta(t) - \beta = \sum_{k=1}^{\infty} J_k(\beta) \psi_k(t) \text{ a.e.,}$$

$$(c) \quad \psi_k(2^n t) = \psi_{2^n k}(t),$$

$$(d) \quad J_{2^n k}(u) = 2^{-n} J_k(2^n u) = 2^{-n} J_k(u_n),$$

where $u_n = 2^n u - [2^n u]$ and $k > 0, n \geq 0$,

(e) the functions $\{\psi_k(t)\}$, $k \geq 0$, form a complete orthonormal system on $(0, 1)$.

Parts (a), (b), (c), (e) are given in [1], and (d) follows readily from (c) and from the fact that $J_k(m) = 0$ for all integers m .

LEMMA 2. For $0 \leq u, v \leq 1$,

$$\sum_{k=1}^{\infty} J_k(u) J_{2^n k}(v) = 2^{-n} \left\{ \int_0^1 G_n(t) G_{v_n}(t) dt - uv_n \right\}$$

$$= 2^{-n} \{ \min(u, v_n) - uv_n \}.$$

PROOF. By Lemma 1(d),

$$\sum_{k=1}^{\infty} J_k(u) J_{2^n k}(v) = 2^{-n} \sum_{k=1}^{\infty} J_k(u) J_k(v_n).$$

An application of Parseval's identity to the function (b) of Lemma 1, with $\beta = u$ and $\beta = v_n$, yields the required result.

Returning to the computation of σ^2 , we have, for $n \geq 0$,

$$\begin{aligned}
\int_0^1 f(t)f(2^n t)dt &= \int_0^1 ((t))((2^n t))dt + \int_0^1 ((t))\{G_\beta(2^n t) - \beta\}dt \\
&\quad + \int_0^1 ((2^n t))\{G_\beta(t) - \beta\}dt \\
&\quad + \int_0^1 \{G_\beta(t) - \beta\}\{G_\beta(2^n t) - \beta\}dt \\
&= A_n + B_n + C_n + D_n,
\end{aligned}$$

say. Now

$$((2^n t)) = -\frac{1}{4} \sum_{k \geq 0} 2^{-k} \psi_{2^n k}(t),$$

by Lemma 1, (a) and (c). Parseval's identity yields

$$A_n = 2^{-n}/12.$$

Also,

$$G_\beta(2^n t) - \beta = \sum_{k=1}^{\infty} J_k(\beta) \psi_{2^n k}(t),$$

by Lemma 1, (b) and (c). Parseval's identity then gives

$$\begin{aligned}
B_n &= -\frac{2^{-n}}{4} \sum_{s=0}^{\infty} 2^{-s} J_{2^s}(\beta) \\
&= 2^{-n} \int_0^\beta \left\{ -\frac{1}{4} \sum_{s=0}^{\infty} 2^{-s} \psi_{2^s}(t) \right\} dt \\
&= 2^{-n} \int_0^\beta ((t))dt = 2^{-n} \left(\frac{\beta^2 - \beta}{2} \right).
\end{aligned}$$

Another application of Parseval's identity yields

$$C_n = -\frac{1}{4} \sum_{k=0}^{\infty} 2^{-k} J_{2^{n+k}}(\beta).$$

By (d) of Lemma 1,

$$\begin{aligned}
C_n &= -\frac{2^{-n}}{4} \sum_{k=0}^{\infty} 2^{-k} J_{2^k}(\beta_n) \\
&= 2^{-n} \left(\frac{\beta_n^2 - \beta_n}{2} \right),
\end{aligned}$$

as above. A final application of Parseval's identity yields

$$\begin{aligned} D_n &= \sum_{k=1}^{\infty} J_k(\beta) J_{2^n k}(\beta) \\ &= 2^{-n} \{ \min(\beta, \beta_n) - \beta \beta_n \}, \end{aligned}$$

by Lemma 2. Collecting our results and substituting in (4a), we obtain

$$\sigma^2(\beta) = \left(\beta - \frac{1}{2} \right)^2 + 2 \sum_{n=1}^{\infty} 2^{-n} \left\{ \frac{\beta_n^2 - \beta_n}{2} + \min(\beta, \beta_n) - \beta \beta_n \right\}.$$

Using the fact that $2 \min(x, y) = x + y - |x - y|$, and performing some elementary manipulations, we have

$$\sigma^2(\beta) = \sum_{n=1}^{\infty} 2^{-n} \left(\frac{1}{2} - |\beta - \beta_n| \right)^2.$$

But $0 \leq |\beta - \beta_n| < 1$, so we may write

$$(7) \quad \sigma^2(\beta) = \sum_{n=1}^{\infty} 2^{-n} ((\beta - \beta_n))^2.$$

Since $((u)) = 0$ only if $u \equiv 1/2 \pmod{1}$, we see that $\sigma^2(\beta) = 0$ if and only if $\beta = 1/2$.

THEOREM 2. *If $f(t) = ((t - \beta))$, $0 \leq \beta < 1$, $\beta \neq 1/2$, then the measure of the set in $(0, 1)$ on which $S_N(f; t) < b$ approaches, as $N \rightarrow \infty$, the integral (5), where $\sigma^2 = \sigma^2(\beta)$ is given by (7), and $\sigma \neq 0$.*

We observe that (7) may be written as

$$(8) \quad \sigma^2(\beta) = \sum_{n=1}^{\infty} 2^{-n} ((\{2^n - 1\}\beta))^2.$$

This form shows that $\sigma^2(\beta)$ is continuous. It appears likely that it is nowhere differentiable, but we have not quite been able to prove this.

For the sake of completeness, we derive the limiting distribution of

$$U_N(t) = N^{1/2} S_N(f; t)$$

for $f(t) = ((t - 1/2))$. Since $f(t) = t_1 - t_0$, $f(2^n t) = t_{n+1} - t_n$, the sum collapses to $U_N(t) = t_N - t_0$. Let $F_N(x)$ denote the cumulative distribution function of $U_N(t)$ in the interval $0 \leq t < 1$,

$$F_N(x) = | \{ t : 0 \leq t < 1, U_N(t) \leq x \} |,$$

where $|E|$ is the Lebesgue measure of the set E . Then $F_N(x) = 1$ for

$x \geq 1$, $F_N(-x) = 1 - F_N(x)$, $F_N(0) = 1/2$. Since $U_N(t)$ is piecewise linear, so is $F_N(x)$, and it is not difficult to see that

$$F'_N(x) = 1 - [2^N x] / (2^N - 1)$$

for $0 < x < 1$, provided that x is not a dyadic rational. Thus $F'_N(x) \rightarrow 1 - x$ uniformly for such values of x , and

$$F_N(x) \rightarrow \frac{1}{2} + \int_0^x (1 - u) du = 1 - \frac{1}{2} (1 - x)^2.$$

Hence the limiting distribution function is

$$(9) \quad F(x) = \begin{cases} 1, & x \geq 1, \\ 1 - \frac{1}{2}(1 - x)^2, & 0 \leq x \leq 1, \end{cases}$$

$$F(-x) = 1 - F(x).$$

3. The functions $\nu_N(t; \alpha, \beta)$. Let $N\nu_N(t; \alpha, \beta)$ be defined as the number of integers n , $0 \leq n \leq N$, such that $t_n = 2^nt - [2^nt]$ falls in the interval $\alpha \leq x < \beta$, where $0 \leq \alpha < \beta \leq 1$. Then

$$N\nu_N(t; \alpha, \beta) = \sum_{n=0}^{N-1} \{G_\beta(2^nt) - G_\alpha(2^nt)\},$$

and the mean value of $\nu_N(t; \alpha, \beta)$ is $\beta - \alpha$. Since $f(t) \equiv G_\beta(t) - G_\alpha(t) - (\beta - \alpha)$ satisfies the first three conditions of (4), we find that

$$\left| \{t: 0 \leq t < 1, N^{1/2}(\nu_N(t; \alpha, \beta) - (\beta - \alpha)) < b\} \right| \rightarrow \Phi(b; \sigma),$$

where σ^2 is given by (4a), provided that $\sigma \neq 0$.

Now

$$f(t) \sim \sum_{k=1}^{\infty} (J_k(\beta) - J_k(\alpha)) \psi_k(t),$$

so by Parseval's identity,

$$\int_0^1 f(t) f(2^nt) dt = \sum_{k=1}^{\infty} (J_k(\beta) - J_k(\alpha)) (J_{2^nk}(\beta) - J_{2^nk}(\alpha)).$$

Applying Lemma 2 four times, we obtain

$$\begin{aligned} & \int_0^1 f(t) f(2^nt) dt \\ &= 2^{-n} \left\{ \int_0^1 (G_\beta(t) - G_\alpha(t)) (G_{\beta_n}(t) - G_{\alpha_n}(t)) dt - (\beta - \alpha)(\beta_n - \alpha_n) \right\}. \end{aligned}$$

Let us write $E = \{t: \alpha \leq t < \beta\}$, $E_n = \{t: t \text{ lies between } \alpha_n \text{ and } \beta_n\}$. Then

$$\int_0^1 f(t)f(2^n t)dt = 2^{-n} \{ |E \cap E_n| - |E| \cdot |E_n| \} \operatorname{sgn}(\beta_n - \alpha_n).$$

Thus we obtain

$$(10) \quad \sigma^2(\alpha, \beta) = |E| - |E|^2 + 2 \sum_{n=1}^{\infty} 2^{-n} \operatorname{sgn}(\beta_n - \alpha_n) \{ |E \cap E_n| - |E| \cdot |E_n| \}.$$

For $\alpha=0$, this becomes

$$(11) \quad \sigma^2(0, \beta) = \beta - \beta^2 + 2 \sum_{n=1}^{\infty} 2^{-n} \{ \min(\beta, \beta_n) - \beta \beta_n \}.$$

It is clear that $\sigma^2(0, \beta)=0$ only if $\beta=0$ or 1. Whether $\sigma^2(\alpha, \beta)$ can vanish for any but the trivial cases does not appear to be an easy question.

4. Joint distributions. Let $f(t)$, $g(t)$ satisfy conditions (4), and let

$$E_N(x, y) = \{ t: 0 \leq t < 1, S_N(f; t) \leq x, S_N(g; t) \leq y \}.$$

We shall show that $\lim_{N \rightarrow \infty} |E_N(x, y)|$ exists and is the distribution function for a two-dimensional normal distribution, and shall give formulas for the parameters of this distribution.

LEMMA 3. *Let (X_N, Y_N) be a sequence of two-dimensional random variables, with probability functions P_N , such that $P_N(h) \rightarrow P(h)$ for every half-space*

$$h: uX + vY \leq t,$$

where P is also a probability function. Then $P_N(I) \rightarrow P(I)$ on all continuity intervals I of P .

PROOF. Let $\phi_N(u, v)$, $\phi(u, v)$, denote the characteristic functions (c.f.'s) corresponding to P_N and P respectively. By the continuity theorem for c.f.'s, it is sufficient to show that $\phi_N(u, v) \rightarrow \phi(u, v)$. Now the distributions of the one-dimensional variables $uX_N + vY_N$ converge to that of $uX + vY$, where (X, Y) has the probability function P , by assumption. Hence

$$\text{c.f. } (uX_N + vY_N) \rightarrow \text{c.f. } (uX + vY),$$

that is,

$$E(\exp(i t(uX_N + vY_N))) \rightarrow E(\exp(i t(uX + vY))).$$

But this means that

$$\phi_N(ut, vt) \rightarrow \phi(ut, vt).$$

Putting $t=1$, we have the desired result.

It is obvious that Lemma 3 can be generalized to k dimensions.

Suppose, now, that f, g satisfy (4). Then so does $h(t) = uf(t) + vg(t)$, in general, and $S_N(h; t)$ is asymptotically normal, with variance

$$(12) \quad \sigma^2 = u^2 \sigma_1^2 + 2ruv\sigma_1\sigma_2 + v^2\sigma_2^2,$$

where σ_1, σ_2 correspond to f, g , respectively, and

$$(13) \quad r\sigma_1\sigma_2 = \int_0^1 f(t)g(t)dt + \sum_{n=1}^{\infty} \left\{ \int_0^1 f(t)g(2^n t)dt + \int_0^1 f(2^n t)g(t)dt \right\}.$$

The only exceptional cases are $r = \pm 1$, with $u\sigma_1 + rv\sigma_2 = 0$. Leaving these aside, let us consider the proper normal variable (X, Y) with mean $(0, 0)$, variances σ_1^2, σ_2^2 , and correlation coefficient r . Then $uX + vY$ is normal with mean 0 and variance σ^2 given by (12). Hence, applying Lemma 3, we have the following:

THEOREM 3. *If f, g satisfy (4), and if r , given by (13), is not ± 1 , then the joint distribution of $S_N(f; t), S_N(g; t)$ is asymptotically normal with mean $(0, 0)$, variances σ_1^2, σ_2^2 , and correlation coefficient r .*

It is not difficult to see that the result of Theorem 3 is still valid if $r = \pm 1$. In this case, all the mass is distributed along a line in the (X, Y) plane.

It may be of interest to note that, corresponding to every f satisfying (4), there is an $f^* \in L_2$ which exhibits the asymptotic behavior of $S_N(f; t)$. Indeed, if

$$(14) \quad f(t) \sim \sum_{k=1}^{\infty} c_k \psi_k(t),$$

we define

$$(15) \quad f^*(t) \sim \sum_{\omega \text{ odd}} C_{\omega} \psi_{\omega}(t),$$

where

$$(16) \quad C_{\omega} = \sum_{n \geq 0} c_{2^n \omega} \quad (\omega = 1, 3, 5, 7, \dots).$$

It is easily proved that the asymptotic variance of $S_N(f; t)$ is precisely

the ordinary variance of $f^*(t)$,

$$(17) \quad \sigma^2 = \sum_{\omega} C_{\omega}^2.$$

Furthermore, the limiting correlation coefficient of $(S_N(f; t), S_N(g; t))$ is the ordinary correlation coefficient of $(f^*(t), g^*(t))$:

$$(18) \quad r = \frac{1}{\sigma_1 \sigma_2} \int_0^1 f^*(t) g^*(t) dt = \sum C_{\omega} C'_{\omega}.$$

Thus, orthogonality of f^* , g^* is equivalent to asymptotic independence of $S_N(f; t)$, $S_N(g; t)$.

5. Two applications. Let us now apply Theorem 3 to the functions $f(t) = G_{\beta}(t) - \beta$, $g(t) = G_{\beta'}(t) - \beta'$. The variances $\sigma^2(0, \beta)$, $\sigma^2(0, \beta')$ are given by (11). Now

$$\int_0^1 f(t) g(2^n t) dt = \sum_{k=1}^{\infty} J_{2^k}(\beta) J_k(\beta') = 2^{-n} \{ \min(\beta_n, \beta') - \beta_n \beta' \},$$

by Lemma 2. Interchanging β and β' , and substituting in (13), we get

$$(19) \quad r\sigma(0, \beta)\sigma(0, \beta') = \min(\beta, \beta') - \beta\beta' \\ + \sum_{n \geq 1} 2^{-n} \{ \min(\beta, \beta'_n) + \min(\beta', \beta_n) - \beta\beta'_n - \beta'\beta_n \}.$$

The right side vanishes only in the trivial cases.

In general, corresponding to the pair of intervals (α, β) , (α', β') , the asymptotic joint distribution of

$$N^{1/2}(\nu_N(t; \alpha, \beta) - (\beta - \alpha)), \\ N^{1/2}(\nu_N(t; \alpha', \beta') - (\beta' - \alpha'))$$

has a correlation coefficient given by

$$(20) \quad r\sigma(\alpha, \beta)\sigma(\alpha', \beta') = |E \cap E'| - |E| \cdot |E'| \\ + \sum_{n=1}^{\infty} 2^{-n} \{ \operatorname{sgn}(\beta_n - \alpha_n)(|E_n \cap E'| - |E_n| \cdot |E'|) \\ + \operatorname{sgn}(\beta'_n - \alpha'_n)(|E'_n \cap E| - |E'_n| \cdot |E|) \},$$

where $\sigma(\alpha, \beta)$, E , E_n have the meanings assigned in §3. We shall omit the proof, which is a straightforward application of Theorem 3 and of Lemma 2. It would be of interest to determine whether there are any nontrivial cases for which $r=0$ or ± 1 .

Finally, for the sums

$$N^{-1/2} \sum_{n=0}^{N-1} ((2^n t - \beta)), \quad N^{-1/2} \sum_{n=0}^{N-1} ((2^n t - \beta')),$$

the asymptotic joint distribution has a correlation coefficient given by

$$(21) \quad \begin{aligned} 2r\sigma(\beta)\sigma(\beta') &= ((\beta - \beta'))^2 \\ &- \frac{1}{4} + \sum_{n=1}^{\infty} 2^{-n} \{ ((\beta' - \beta_n))^2 + ((\beta - \beta'_n))^2 \}, \end{aligned}$$

where $\sigma(\beta)$ is given by (7). The calculations are standard by now, and we omit the proof.

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