ON THE METRIZABILITY OF THE BUNDLE SPACE

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It has been shown by Yu. M. Smirnov (see [**1**, p. 13, Theorem 3]) that a Hausdorff space X is metrizable if and only if X is paracompact and has an open cover each of whose members is metrizable.

Using this, we prove: *If* $\{X, B, \pi, Y, \mathcal{V}, \phi, G\}$ *is a fibre bundle* (see [**2**, p. 7]) *whose base space B and fibre Y are metrizable, then the bundle space X is also metrizable.*

First, if B and Y are metrizable, then X is Hausdorff and has an open cover each of whose members is metrizable, namely $\{\pi^{-1}(U) \mid U \in \mathcal{V}\}$, since if $U \in \mathcal{V}$ then $U \times Y$, and hence $\pi^{-1}(U)$, is metrizable.

Second, X is paracompact, for let $\mathcal{C}$ be any open cover for X. Let $\mathcal{G}$ be a locally finite refinement of $\mathcal{V}$, and let $\mathcal{W}$ be a closure refinement of $\mathcal{G}$ which is also locally finite; define $\lambda: \mathcal{W} \to \mathcal{G}$ a function such that, for each W, if $W \in \mathcal{W}$ then $W^- \subset \lambda(W)$. Let for each $V \in \mathcal{G}$

$$\mathcal{D}_V = \{C \cap \pi^{-1}(V) \mid C \in \mathcal{C}\}$$

and let $\mathcal{D}_V^*$ be a locally finite refinement of $\mathcal{D}_V$ (this exists, since $\pi^{-1}(V)$ is metric, hence paracompact, for each $V \in \mathcal{G}$). For $W \in \mathcal{W}$ define

$$\mathcal{E}_W = \{D \cap \pi^{-1}(W) \mid D \in \mathcal{D}^*_{\lambda(W)}\}$$

and let $\mathcal{J} = \bigcup_{W \in \mathcal{W}} \mathcal{E}_W$. Clearly $\mathcal{J}$ refines $\mathcal{C}$, and it is thus sufficient to show that $\mathcal{J}$ is locally finite.

Let $x \in X$; then since $\mathcal{W}$ is locally finite, so is $\{\pi^{-1}(W) \mid W \in \mathcal{W}\}$; therefore there is a neighborhood A of x which meets only a finite number of members of $\{\pi^{-1}(W) \mid W \in \mathcal{W}\}$, say $\pi^{-1}(W_1), \cdots, \pi^{-1}(W_n)$. For $1 \leqq i \leqq n$ define B_i a neighborhood of x as follows:

(1) if x is a member of $\pi^{-1}(W_i)$ let B_i be a neighborhood of x which meets only a finite number of members of $\mathcal{E}_{W_i}$;

(2) if x is a member of the boundary of $\pi^{-1}(W_i)$ let B_i be a neighborhood of x which meets only a finite number of members of $\mathcal{D}^*_{\lambda(W_i)}$, hence only a finite number of members of $\mathcal{E}_{W_i}$;

(3) if x is not in the closure of $\pi^{-1}(W_i)$ let B_i be a neighborhood of

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x which does not meet $\pi^{-1}(W_i)$, hence which meets no members of $\mathcal{E}_{W_i}$.

Then $A \cap \bigcap_{i=1}^{n} B_i$ is a neighborhood of x which meets only a finite number of members of $\mathcal{J}$, since it meets only members of $\mathcal{E}_{W_1}, \cdots, \mathcal{E}_{W_n}$, and at most a finite number of each of these.

References

1. Yu. M. Smirnov, *On the metrization of topological spaces*, Amer. Math. Soc. Translation, No. 91.

2. N. E. Steenrod, *The topology of fibre bundles*, Princeton University Press, 1951.

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