FIXED POINTS OF PRODUCTS AND ORDERED SUMS OF SIMPLY ORDERED SETS

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Let A and B be simply ordered sets and $A \times B$ the cartesian product ordered by first differences. Sufficiency conditions are given on the sets A and B for the existence of fixed points in the set $A \times B$ (Theorem 2). Sufficiency conditions are given on the sets A , B , and C for the incomparability of the order types¹ $|A \times B|$ and $|A \times C|$ ($|C \times A|$ and $|C \times B|$) (Corollary 2 of Theorem 7). The content of Theorem 4 is that if $\{A_\xi\}$ is a family of $2^{\aleph_0}$ linear sets, each containing a fixed point, then there exists a subset G of $R \times R$, where R denotes the real numbers, such that for each element x in R , if $B_x = \{y | (x, y) \in G\}$, either B_x is empty or is one of the sets A_ξ . Furthermore, if p is a fixed point in B_x , then (x, p) is a fixed point in G .

1. Fixed points of ordered sums and products. The element p of the simply ordered set A is said to be a fixed point of A if $f(p) = p$ for every similarity transformation f of A into A .

LEMMA 1. *Let p and q be fixed points of the disjoint simply ordered sets A and B respectively. Then either p or q is a fixed point of the ordered sum, $C = A + B$.²*

PROOF. Assume the contrary, i.e., assume that there exist similarity transformations f and g of C into C for which $f(p) \neq p$ and $g(q) \neq q$. Suppose that $f(p) < p$. Then the function h , which is defined by $h(x) = x$ for $x > p$, x in A , and $h(x) = f(x)$ for $x \leq p$, is a similarity transformation of A into A for which $h(p) \neq p$. From this contradiction of p being a fixed point in A , we see that $f(p) > p$. Analogous arguments show that $g(q) < q$, $f(q) \leq q$, and $g(p) \geq p$. Let k be the similarity transformation of C into C which is defined by $k(x) = f(x)$ for $x < q$ and $k(x) = x$ for $x \geq q$. This shows that without loss of generality we

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¹ If A is a simply ordered set, $|A|$ denotes its order type.

² Let B be a simply ordered set. If $\{A_b | b \in B\}$ is a family of pairwise disjoint simply ordered sets, then the ordered sum, $\sum_{b \in B} A_b$, is the set union of the A_b , the elements being ordered as follows. If c and d are two distinct elements in A_b , then c and d , considered in the ordered sum, have the same order as in the set A_b . If c is in A_a and d is in A_b , where $a < b$, then $c < d$.

may suppose that $f(x) = x$ for $x \geq q$, and likewise, that $g(x) = x$ for $x \leq p$.

Suppose that $fg(B)$ is a subset of B . As q is a fixed point in B , $fg(q) = q$. On the other hand, since $g(q) < q$, it follows that $fg(q) < f(q) = q$. From this contradiction we conclude that $fg(B) \cap A$ is nonempty, and thus that $fg(A)$ is a proper subset of A .

Since fg , acting on A , is a similarity transformation of A into A , $fg(p) = p$. But $f[g(p)] = f(p) > p$. This is a contradiction. Thus either p or q is a fixed point in C . Q.E.D.

REMARK. It is not true, in general, that both p and q are fixed points in C . One need only consider the case where $A = \{\xi \mid \xi \leq \omega\}$, $B = \{\xi \mid \omega < \xi \leq \omega + \omega\}$, $p = \omega$, and $q = \omega + \omega$.

COROLLARY. If $\{A_1, A_2, \dots, A_n\}$ are a family of pairwise disjoint sets, and p_j is a fixed point in A_j , then at least one of the points p_j is a fixed point in the ordered sum, $C = A_1 + \dots + A_n$.

THEOREM 1. Let $\{A_1, A_2, \dots, A_n\}$ be a family of pairwise disjoint sets such that³ $|A_i| = |A_j|$ for $i, j \leq n$. For $i = 2, 3, \dots, n$, let f_i be a similarity transformation of A_1 into A_i . Let p be a fixed point of A_1 , and B the ordered sum, $A_1 + A_2 + \dots + A_n$. (a) If p is a fixed point in $A_1 \cup A_2$, then p is a fixed point in B , and thus a fixed point in $\bigcup_{i \leq j} A_i$. If $f_2(p)$ is a fixed point in $A_1 \cup A_2$, then $f_n(p)$ is a fixed point in B . (b) If both p and $f_2(p)$ are fixed points in $A_1 \cup A_2$, then each point $f_i(p) = p_i$ is a fixed point in B .

PROOF. Using induction suppose that the theorem has been demonstrated for $n = 2, 3, \dots, k$.

To prove the first sentence in (a) suppose that p is not a fixed point in the set $B = \bigcup_{i \leq k+1} A_i$, i.e., there is a similarity transformation f of B into B for which $f(p) \neq p$. Let m be the largest integer $\leq k$ for which $f(p_j) \neq p_j$ for $j \leq m$. If $m < k$, then $f(p_{m+1}) = p_{m+1}$. Thus the function g on $C = \bigcup_{i \leq k} A_i$, which is defined by $g(x) = f(x)$ for $x < p_{m+1}$ and $g(x) = x$ for $x \geq p_{m+1}$, is a similarity transformation. Since $g(p) \neq p$, our induction hypothesis that p is a fixed point in C is contradicted. Now suppose that $m = k$. As is easily seen from our induction hypothesis, $f(p_m) > p_m$. Let $D = A_k \cup A_{k+1}$. Let h be the similarity transformation of D into D which is defined by $g(x) = x$ for $x < p_m$, and $g(x) = f(x)$ for $x \geq p_m$. Since $|A_i| = |A_j|$, from I of §2 of [4], p_m is a fixed point in D . But $g(p_m) \neq p_m$. From this contradiction we see that p must be a fixed point in B .

³ $|A_i| = |A_j|$ if there exists a similarity transformation of A_i into A_j , and a similarity transformation of A_j into A_i .

To see the second sentence in (a) let g_2 be a similarity transformation of A_2 into A_1 . Since $g_2 f_2$ is a similarity transformation of A_1 into A_1 and p is a fixed point in A_1 , $g_2[f_2(p)] = p$. Let σ be the similarity transformation of $A_1 \cup A_2$ into $A_k \cup A_{k+1}$ which is defined by $\sigma(x) = f_k(x)$ for x in A_1 and $\sigma(x) = f_{k+1}g_2(x)$ for x in A_2 . Since $|A_1 \cup A_2| \equiv |A_k \cup A_{k+1}|$, by I of §2 of [4], $\sigma[f_2(p)] = f_{k+1}(p)$ is a fixed point in $A_k \cup A_{k+1}$. By a method analogous to that in the preceding paragraph one shows that p_{k+1} is fixed in $A_{k-1} \cup A_k \cup A_{k+1}$, then fixed in $A_{k-2} \cup A_{k-1} \cup A_k \cup A_{k+1}$, etc., and finally, fixed in $\bigcup_{j \leq k+1} A_j$.

To see part (b) suppose that both p and p_2 are fixed points in the set $A_1 \cup A_2$. Furthermore, suppose that for some j , there exists a similarity transformation f of B into B for which $f(p_j) \neq p_j$. Clearly $j < n$. If $f(p_j) < p_j$, then it is easy to construct a similarity transformation g of $\bigcup_{i \leq j} A_i$ into $\bigcup_{i \leq j} A_i$ for which $g(p_j) \neq p_j$. This contradicts part (a) for $n = j$. If $f(p_j) > p_j$, then there exists a similarity transformation h of $\bigcup_{j \leq i \leq n} A_i$ into $\bigcup_{j \leq i \leq n} A_i$ for which $h(p_j) \neq p_j$. By proceeding as in the previous paragraph we see that p_j is fixed in $A_j \cup A_{j+1}$. On applying (a) we see that p_j is fixed in $\bigcup_{j \leq i \leq n} A_i$. Since $h(p_j) \neq p_j$ we have a contradiction. Therefore $f(p_j) = p_j$. Q.E.D.

We now consider the existence of fixed points in the set $A \times B$, where $A \times B$ is ordered by first differences, i.e., $(a, b) < (c, d)$ if $a < c$, or if $a = c$ and $b < d$. It is obvious that if either A or B contains no fixed points, then $A \times B$ contains none. The question arises as to what occurs in the case where A and B both contain fixed points.

THEOREM 2. *Let p and q be fixed points respectively of the simply ordered sets A and B . Then there exists a fixed point (r, q) of $A \times B$, where r is some fixed point of A such that both of the sets*

$$\{x \mid r \leq x \leq p, x \in A\} \quad \text{and} \quad \{x \mid p \leq x \leq r, x \in A\}$$

are finite.

PROOF. For two points a and b in A let aRb if both of the sets

$$\{x \mid a \leq x \leq b, x \in A\} \quad \text{and} \quad \{x \mid b \leq x \leq a, x \in A\}$$

are finite. Let N_a be the equivalence class, determined by the equivalence relation R , which contains the element a . The order type of the set N_a is either ω , ω^* , $\omega^* + \omega$, or n , where n is finite.

Now let f be a similarity transformation of $A \times B$ into $A \times B$. Suppose that the order type of the set N_a is ω , i.e., the elements of N_a are $y_1 < y_2 < \dots < y_n < \dots$. Let E be the set of A coordinates of the elements of the set $f(N_a \times B)$. Since q is a fixed point of B , by Lemma

1.3 of [4]⁴, $|m \times B| < |(m+1) \times B|$ for each positive integer m . Thus the set E is infinite. Let $z_1 < z_2 < \dots < z_n < \dots$ be a sequence of points of E . Let $g(y_n) = z_{4n+2}$ if $y_n \neq z_{4n+2}$, and $g(y_n) = z_{4n+3}$ if $y_n = z_{4n+2}$. If the order type of the set N_a is ω^* , or $\omega^* + \omega$, then g can be defined analogously to the above, so that $g(x) \neq x$ for each x in N_a . Finally, suppose that the order type of the set N_a is n , where n is finite. Again let E be the set of A coordinates of elements in the set $f(N_a \times B)$. Let $F = \{x | c < x < d, c \in E, d \in E, x \in A\}$ and $G = E \cup F$. Since $|B| |j| < |B| |(j+1)|$ the power of the set G is at least n . If the power of the set G is greater than n , then a similarity transformation g , of N_a into G , can be defined in such a manner that $g(x) \neq x$. If the power of G is exactly n , then a similarity transformation g , from N_a to G , can be defined in precisely one way.

The function g is thus defined to be one-to-one on each set N_a , and as is easily seen, is also one-to-one on all of A . Clearly g preserves the order of the elements. Thus g is a similarity transformation of A into A . As p is a fixed point of A , $g(p) = p$. This implies that N_p is finite and the set G that is associated with N_p contains exactly as many elements as N_p . Since each element y in N_p is also a fixed point in A (IV of §2 of [4]), $g(y) = y$. This shows that $f(N_p \times B)$ is a subset of $N_p \times B$. This is also true for any other similarity transformation h of $A \times B$ into $A \times B$. Since $N_p \times B$ is the ordered sum of n disjoint sets which are similar to B , n being the number of elements in N_p , by the Corollary to Lemma 1, one of the points (r, q) , where r is some element in N_p , is a fixed point. Q.E.D.

If each point of B is a fixed point of B , then the conclusion of Theorem 2 can be strengthened. First, however, we prove a general result on similarity transformations.

LEMMA 2. Let g be a similarity transformation of $C \times D$ into $E \times F$ which has the property that for each element a in C , there are at least two elements $u(a)$ and $v(a)$, $u(a) < v(a)$, in E such that

$$g(a \times D) \cap (u(a) \times F) \neq \emptyset \quad \text{and} \quad g(a \times D) \cap (v(a) \times F) \neq \emptyset.$$

Then (a) for a given element p in C and a given element q in E , where $u(p) \leq q \leq v(p)$, a similarity transformation h of C into E can be found so that $h(p) \neq q$, and for each element a in C , either $h(a) = u(a)$ or $h(a) = v(a)$; and (b) if $C = E$, then h can be chosen so that $h(a) \neq a$ and $u(a) \leq h(a) \leq v(a)$.

PROOF. Let $[u(a), v(a)] = \{x | u(a) \leq x \leq v(a), x \in E\}$, and W

⁴ $|A| < |B|$ if there exists a similarity transformation of A into B , but no similarity transformation of B into A . See [7, p. 253].

$= \{ [u(a), v(a)] \mid a \in C \}$. Since g is a similarity transformation, it follows that no element of E can be in more than two elements of W . Furthermore, if the element x is in two elements of W , say w_1 and w_2 , then $w_1 = [a, v]$ and $w_2 = [z, a]$, or $w_1 = [z, a]$ and $w_2 = [a, v]$. Clearly if $a < b$, then $u(a) < v(a) \leq u(b) < v(b)$.

Suppose that the two elements, p in C and q in E , $u(p) \leq q \leq v(p)$, are given. If $u(p) \neq q$, let $h(x) = u(x)$ for x in C . If $u(p) = q$, let $h(x) = v(x)$ for x in C . The function h satisfies the conclusions in (a).

Now suppose that $C = E$. For any two elements a and b in E , let aRb and N_a have the same meaning as in Theorem 2. Suppose that the order type of N_a is ω , the elements of N_a being $y_0 < y_1 < \dots$. It follows from the definition of the set N_a and the fact that the y_n are a sequence of consecutive elements in E that $[u(y), v(y)] \cap [u(y_n), v(y_n)] = \emptyset$ for any element y in $E - N_a$. Let $h(y_0) = u(y_0)$ if $u(y_0) \neq y_0$, and $h(y_0) = v(y_0)$ if $u(y_0) = y_0$. Suppose that $h(y_i)$ has been defined for $i \leq j$. Two possibilities occur. Either $v(y_j) < u(y_{j+1})$ or $v(y_j) = u(y_{j+1})$. Consider the first alternative. Let $h(y_{j+1}) = u(y_{j+1})$ if $u(y_{j+1}) \neq y_{j+1}$, and $h(y_{j+1}) = v(y_{j+1})$ if $u(y_{j+1}) = y_{j+1}$. Consider the second alternative. If $v(y_{j+1}) \neq y_{j+1}$, let $h(y_{j+1}) = v(y_{j+1})$. Suppose that $v(y_{j+1}) = y_{j+1}$. As y_j and y_{j+1} are consecutive elements in E , $v(y_j) \leq y_j$. If $v(y_j) = y_j$, then from our induction hypothesis that $h(y_j) \neq y_j$, and $h(y_j) \leq v(y_j)$, we see that $h(y_j) < y_j$. In this case, let $h(y_{j+1}) = u(y_{j+1}) = y_j$. If $v(y_j) < y_j$, let $h(y_{j+1}) = y_j$. Note that $u(y_{j+1}) \leq h(y_{j+1}) \leq v(y_{j+1})$. In this way the function h is defined on N_a . If the order type of N_a is ω^* , $\omega^* + \omega$, or n , then an analogous argument is applicable. The function h is thus defined on each set N_a , and hence on all of E . Clearly it is a similarity transformation and satisfies the conclusions in (b). Q.E.D.

A simply ordered set E is called *exact* if E is nonempty and if each point of E is a fixed point.

THEOREM 3. *Let p be a fixed point of E and let F be an exact set. Then for each element y in F , (p, y) is a fixed point of $E \times F$.*

PROOF. Let g be any similarity transformation of $E \times F$ into itself. Let a be any element of E for which there exists an element b such that $g(a \times F) \subseteq (b \times F)$. For each element y in F , let $m(y) = z$, where $g[(a, y)] = (b, z)$. The function $m(y)$ is a similarity transformation of F into F . As F is exact, $m(y) = y$ for each element y in F , i.e., $g[(a, y)] = (b, y)$. Therefore $g(a \times F) = (b \times F)$.

Suppose that $g(p \times F)$ is not the set $(p \times F)$. Let

$$B = \{ a \mid a \in E, g(a \times F) = (b_a \times F) \} \quad \text{and} \quad G = \{ b_a \mid a \in B \}.$$

Let $H = E - B$ and $M = E - G$. Since $g(a \times F) = (b_a \times F)$ for each element a in B , it follows that $g(H \times F) \subseteq (M \times F)$. Furthermore, for each element y in H , there are two elements, $u(y)$ and $v(y)$, $u(y) < v(y)$, in M such that

$$g(y \times F) \cap (u(y) \times F) \neq \emptyset \quad \text{and} \quad g(y \times F) \cap (v(y) \times F) \neq \emptyset.$$

Two possibilities arise. Either p is in B or p is in H . Suppose that the first alternative occurs. Then $g(p \times F) = (b_p \times F)$, where $p \neq b_p$. An application of Lemma 2 yields a similarity transformation h of H into M such that $u(x) \leq h(x) \leq v(x)$, and $h(x) \neq x$. Let f be the function which is defined by $f(x) = h(x)$ for each element x in H , and $f(a) = b_a$ for each element a in B . The function f is a similarity transformation of E into E for which $f(p) \neq p$. This is a contradiction of the fact that p is a fixed point of E . Suppose that the second possibility occurs, i.e., p is in H . Applying Lemma 2 we obtain a function h of H into M for which $h(p) \neq p$, and $u(x) \leq h(x) \leq v(x)$ for each element x in H . Let $f(x) = h(x)$ for x in H and $f(a) = b_a$ for a in B . Again $f(p) \neq p$. Thus both alternatives lead to a contradiction. Consequently we must have $g(p \times F) = (p \times F)$, i.e., $g[(p, y)] = (p, y)$ for each element y in F . Therefore (p, y) is a fixed point of $E \times F$.

COROLLARY. *If E and F are both exact, then $E \times F$ is exact.*

2. Imbedding sets. The set of real numbers, ordered in the natural manner, is denoted by R . A subset of R is called a linear set. By θ is meant the smallest ordinal number whose power is $2^{\aleph_0}$.

Let A_0 be a given linear exact set of power $2^{\aleph_0}$. For each ordinal number ξ where $1 \leq \xi < \theta$ let A_ξ be a linear set which is similar to A_0 . Well order the elements of A_0 into the sequence $\{x_\xi\}_{\xi < \theta}$ and let $G = \bigcup_{\xi < \theta} (x_\xi \times A_\xi)$. Consider the set G so obtained. G is a subset of $R \times R$. The order type of G is $A_0 \times A_0$, and so is exact by the corollary to Theorem 3. Furthermore, for each element x in R , if $B_x = \{y \mid (x, y) \in G\}$, then either B_x is empty or else is one of the sets A_ξ . This suggests the question of whether or not there exists an exact subset G of $R \times R$, which has the property that the family of sets $\{B_x \mid x \in R\}$ contains all linear exact sets. No such set G can exist, however, since there are $2^{2^{\aleph_0}}$ exact linear sets and only $2^{\aleph_0}$ sets B_x . Now if we only demand that the family of sets $\{B_x\}$ contain a pre-assigned family of $2^{\aleph_0}$ exact linear sets, then the set G does exist. More precisely we have

THEOREM 4. *Let $\{A_\xi \mid \xi < \theta\}$ be a family of nonempty linear sets A_ξ such that for each ξ , $|A_\xi| < |R|$. Then there exists a subset G of $R \times R$ with the following properties:*

- (a) For each element x in R , the set $B_x = \{y \mid (x, y) \in G\}$ is either empty or else is one of the sets A_ξ .
- (b) For each set A_ξ there is only one element x in R such that $B_x = A_\xi$.
- (c) The set $T = \{x \mid B_x \neq \emptyset\}$ is exact.
- (d) If p is a fixed point in B_x , then (x, p) is a fixed point in G .

PROOF. The demonstration is divided into two cases.

(a) Suppose that only a denumerable number of the sets, say $\{A_n \mid n < \gamma \leq \omega\}$, are of power ≥ 2 each. By (a, b) is meant the open interval $\{x \mid a < x < b\}$ of R . Let S denote the set of positive real numbers. For each $n < \gamma$ let C_n be a subset of $(2n, 2n+1)$ which is similar to A_n . Let f be a similarity transformation of S into S which is not the identity function on $(2n+1, 2n+2)$ for some $n < \gamma$. Let F be the family of all such functions f . The power of F is $2^{\aleph_0}$. Denote by f^* the inverse function of f . Well order the elements of S and F into the two sequences, $\{y_\xi\}_{\xi < \theta}$ and $\{f_\xi\}_{\xi < \theta}$, respectively. For each ξ let

$$D_\xi^n = \{x \mid f_\xi(x) \neq x, x \in (2n+1, 2n+2)\}$$

and $D_\xi = \bigcup_{n < \gamma} D_\xi^n$. Clearly each set D_ξ contains some open interval of S , call it J_ξ . Since $|A_n| = |C_n| < |R|$, by Theorem 8 of [7], $\sum_{n < \gamma} |C_n| < |R|$. Let $Z = \bigcup_{n < \gamma} C_n$, p_0 be the first element in the set $J_0 - f_0^*(Z)$, and $q_0 = f_0(p_0)$. The element p_0 certainly exists since, by Lemmas 3 and 4 of [7], if $|A| < |R|$ and $|B| \equiv |R|$, then the power of the set $B - A$ is $2^{\aleph_0}$. Now suppose that $\{p_\xi\}_{\xi < \mu}$ and $\{q_\xi\}_{\xi < \mu}$ where $\mu < \theta$ have been defined. If $G_\mu = Z \cup E_\mu$, where

$$E_\mu = \{p_\xi \mid \xi < \mu\} \cup \{q_\xi \mid \xi < \mu\},$$

then by Lemma 3 of [7], $|f_\mu^*(G_\mu) \cup E_\mu| < \bar{R}$. Let p_μ be the first element in the set $J_\mu - [f_\mu^*(G_\mu) \cup E_\mu]$ and $q_\mu = f_\mu(p_\mu)$. Let $Y = Z \cup \{p_\xi \mid \xi < \theta\}$.

Suppose that p is either a fixed point in C_i or p is a point in $Y - Z$, which is not a fixed point in Y . Then there exists a similarity transformation g of Y into Y such that $g(p) \neq p$. Either p is in some set C_i , where $i < \gamma$, or else p is in some set $(2n+1, 2n+2)$, where $n < \gamma$. Assume that the former is the situation. Since p is a fixed point of C_i , $g(C_i)$ is not a subset of C_i . This implies that there exists an element q in Y such that $g(q) \neq q$ and q is in either $(2i-1, 2i)$ or $(2i+1, 2i+2)$. From this we see that g is not the identity function on $Y \cap (\bigcup_{n < \gamma} (2n+1, 2n+2))$. By a familiar argument, such as given in Theorem 3 of [2] and Theorem 2.4 in [4], it can be shown that each set $Y \cap (2n+1, 2n+2)$, for $n < \gamma$, is a dense exact subset of $(2n+1, 2n+2)$. From this it follows that the function g can be extended to become an element f_v of F . Therefore the element $g(p_v) = q_v$ is not in

Y , so that g does not map Y into Y . From this contradiction we conclude that the point p is fixed in Y .

A similar argument shows that if $W = \{p_\xi | \xi < \theta\} \cup \{w_n | n < \gamma\}$, where w_n is some definite point in C_n , then each point of W is fixed, i.e., W is exact.

Now the power of the set $Y - Z$ is $2^{\aleph_0}$. Well order the elements of the set $Y - Z$ into the sequence $\{x_\xi\}_{\gamma \leq \xi < \theta}$. Denote by G the set

$$G = \bigcup_{n < \gamma} [(2n + \frac{1}{2}) \times A_n] \cap \{(x_\xi \times A_\xi) | \gamma \leq \xi < \theta\}.$$

Evidently $|G| = |Y|$ since each set A_ξ , $\xi \geq \gamma$, consists of one point. It is easy to verify that G satisfies the conclusions of the theorem.

(b) Suppose that a nondenumerable number of the sets A_ξ are of power ≥ 2 each. Let C be a dense exact subset of R , of power $2^{\aleph_0}$, which has the property that no two disjoint subsets of C , of power $2^{\aleph_0}$ each, are similar. Furthermore, let each point of C be a c -condensation point of C , where by x being a c -condensation point of C is meant that each open interval of R , (a, x) and (x, b) , meets the set C in $2^{\aleph_0}$ points. The existence of such a set C is guaranteed by Lemma 1 and Theorems 4 and 5 of [3]. Divide the family of sets $\{A_\xi | \xi < \theta\}$ into two disjoint families, $\{D_\xi | \xi < \mu \leq \theta\}$ and $\{C_\xi | \xi < \theta\}$, where the first family is a nondenumerable number of sets, of power ≥ 2 each. As is well known the elements of C can be divided into two disjoint sets, $M = \{u_\xi | \xi < \mu\}$ and $N = \{v_\nu | \nu < \theta\}$, such that M is dense in C , and each element u_ξ is a condensation point of C (in the usual sense). Now denote by G the set

$$G = \bigcup_{\xi < \mu} (u_\xi \times D_\xi) \cup \bigcup_{\nu < \theta} (v_\nu \times C_\nu).$$

The only property of G that is not self-evident is (d). To see that (d) is satisfied suppose the contrary, i.e., suppose that there exists a similarity transformation of G into G such that $f(p) \neq p$ for some element p in G , where $p = (d, t)$ and t is a fixed point in B_d . Since t is a fixed point in B_d , it follows that for some element (d, q) in the set (d, B_d) , the element of $f[(d, q)]$ is not in $(d \times B_d)$, i.e., $f[(d, q)] = (z, r)$ where $d \neq z$. Suppose that $z > d$. Denote by P the set $P = \{x | d < x < z, x \in C\}$. Note that $f[(x, y)] > (x, y)$ for (x, y) in G , where x is in P . Since each element of C is a c -condensation point of C , the power of P is $2^{\aleph_0}$. Suppose that for two elements x_1 and x_2 , $x_1 < x_2$, in P , there exists an element x_3 in C such that

$$f(x_1 \times B_{x_1}) \cap (x_3 \times B_{x_3}) \neq \emptyset \text{ and } f(x_2 \times B_{x_2}) \cap (x_3 \times B_{x_3}) \neq \emptyset.$$

This implies that for each element u_ξ in $M \cap P$, $f(u_\xi \times D_\xi) \subseteq (x_3 \times B_{x_3})$.

Now there are a nondenumerable number of such elements u_ξ , from the selection of the u_ξ and v_ξ . Since each set D_ξ contains at least two elements, it follows that the set $(x_3 \times B_{x_3})$ contains a nondenumerable number of disjoint intervals. But this is impossible since $(x_3 \times B_{x_3})$ is similar to a subset of R . Consequently, for any two elements x_1 and x_2 in P , $f(x_1 \times B_{x_1})$ and $f(x_2 \times B_{x_2})$ cannot meet the same set $(x_3 \times B_{x_3})$. This fact implies the following:

For each subset $(x \times B_x)$ of G , where x is in P , select a definite element w_x in B_x . Let g be the function which is defined by $g(x) = y$, where $f[(x, w_x)] = (y, t)$. Then g is a similarity transformation of P into $C - P$.

From this we obtain two similar, disjoint subsets of C , of power $2^{\aleph_0}$ each, P and $g(P)$. This, in turn, contradicts the fact that C does not contain two such sets. Therefore $f[(d, q)] > (d, q)$ is false. An analogous argument shows that $f[(d, q)] < (d, q)$ is also false. Hence we are forced to conclude that no such function f can exist, i.e., the original point p is fixed. Q.E.D.

COROLLARY 1. *Let $\{A_\xi\}_{\xi < \theta}$ be a sequence of nonempty linear sets such that $|A_\xi| < |R|$ for each ξ . Then there exists a subset G of $R \times R$ which satisfies (a), (c), and (d) of Theorem 4, and for which there exists a one-to-one correspondence h between the sets $\{\xi \mid \xi < \theta\}$ and the set $\{x \mid B_x \neq \emptyset\}$ such that $A_\xi = B_{h(\xi)}$.*

COROLLARY 2. *Let $\{A_\xi \mid \xi < \mu \leq \theta\}$ be a family of nonempty linear sets such that $|A_\xi| < |R|$ for each ξ . Then there exists a subset G of $R \times R$ which satisfies (a), (c), and (d) of Theorem 4.*

If each set A_ξ contains at least one fixed point, then $|A_\xi| < |R|$. If each set A_ξ is exact we obtain

COROLLARY 3. *Let $\{A_\xi \mid \xi < \theta\}$ be a family of linear exact sets. Then there exists an exact subset G of $R \times R$ satisfying (a), (b), and (c) of Theorem 4.*

The assumption in Theorem 4 that $|A_\xi| < |R|$ for each ξ cannot be removed. For example, let $A_0 = R$ and $A_\xi = \{p_\xi\}$, $\xi \geq 1$, where $p_\xi \neq p_v$ for $\xi \neq v$. Any subset G of $R \times R$ satisfying (a) and (b) is of order type $\equiv |R|$, thus contains no fixed point, and so violates (d).

The conclusion of Corollary 2 cannot be strengthened to include (b). This is seen by the following example. Let A_0 be a linear set containing a fixed point. For each positive integer i let A_i be a linear set, similar to A_0 , such that $A_m \neq A_n$ for $m \neq n$. Now suppose that a subset G of $R \times R$ exists which satisfies (a) and (b). Then the set T is denumerably infinite. But in Theorem 1 of [2] Dushnik and Miller have

demonstrated that there is no denumerably infinite exact set. Thus (c) and (d) are not satisfied.

3. Related results. In this section some applications of Lemma 2 are given.

A simply ordered set E is said to have the fixed point property if, for each similarity transformation f of E into itself, there exists an element p of E such that $f(p) = p$.

The ordered couple of simply ordered sets (E, F) is called an A -pair if, for each pair of simply ordered sets G and H , and similarity transformation f of $E \times G$ into $F \times H$, there exist two elements, p in E and q in F , such that $f(p \times G)$ is a subset of $(q \times H)$ [1].

From Lemma 2 there immediately follows

THEOREM 5. *If A and B are any two sets for which there is no similarity transformation of A into B , then (A, B) is an A -pair.*

Turning to sets E for which (E, E) is an A -pair we have

THEOREM 6. *If E has the fixed point property, then (E, E) is an A -pair.*

PROOF. Let g be a similarity transformation of $E \times D$ into $E \times F$, and suppose that there are no two elements a and b in E for which $g(a \times D)$ is a subset of $(b \times F)$. This means that for each element a in E there are two elements, $u(a)$ and $v(a)$, $u(a) < v(a)$, such that

$$g(a \times D) \cap (u(a) \times F) \neq \emptyset \quad \text{and} \quad g(a \times D) \cap (v(a) \times F) \neq \emptyset.$$

Applying Lemma 2 we obtain a similarity transformation h of E into E such that for each element a in E , $h(a) \neq a$. This contradicts our assumption that E has the fixed point property. It therefore follows that there must exist two elements a and b in E such that $g(a \times D)$ is a subset of $(b \times F)$, i.e., (E, E) is an A -pair.

COROLLARY 1. *If E has a fixed point, then (E, E) is an A -pair.*

Let E be a well ordered set of order type α . Arens has shown that if $\alpha = \omega$, then (E, E) is not an A -pair [1]. We now extend that result. We may suppose that E is the set of ordinal numbers which are smaller than α . Suppose that α is a limit number. Consider the mapping f of $E \times \{0, 1\}$ onto $E \times \{0\}$ which is defined as follows. For $\xi < \alpha$ and $\xi = \sigma_\xi + n_\xi$, where $\sigma_\xi = 0$ or is a limit number, and n_ξ is a finite integer, let $f[(\xi, 0)] = (\sigma_\xi + 2n_\xi, 0)$ and $f[(\xi, 1)] = (\sigma_\xi + 2n_\xi + 1, 0)$. As α is a limit number f is a similarity transformation of $E \times \{0, 1\}$ onto $E \times \{0\}$. As f is one-to-one, there are no two elements a and b in E for which $f(a \times \{0, 1\})$ is a subset of $(b, 0)$. Thus (E, E) is not

an A -pair. Now suppose that α is a nonlimit number. Then the hypotheses of Corollary 1 of Theorem 6 are satisfied, so that (E, E) is an A -pair. Summarizing the preceding discussion we get

COROLLARY 2. *Let E be a well ordered set of order type α . Then (E, E) is an A -pair if and only if α is not a limit number.*

$|A|$ and $|B|$ are said to be incomparable if there is no similarity transformation of A into B and no similarity transformation of B into A . It is easy to find examples in which $|A|$ and $|B|$ are incomparable, while for some nonempty set C , $|A \times C|$ and $|B \times C|$ ($|C \times A|$ and $|C \times B|$) are not. This is so in the case where $|A| = \omega$, $|B| = \omega^*$, and $|C| = \eta$, η being the order type of the rational numbers ordered in the natural manner. In fact, for each pair of nonempty sets A and B , there always exists a set C such that $|A \times C| \equiv |B \times C|$ ($|C \times A| \equiv |C \times B|$). For let $|D| = |A| + |B|$ and $|C| = |D|^\eta$ [5]. Then $|C||C| = |C|$ and $|C| \equiv |A \times C| \equiv |B \times C|$ ($|C| \equiv |C \times A| \equiv |C \times B|$).

THEOREM 7. *Let A and B be any two sets for which there is no similarity transformation of A into B . Then*

(a) *for any set C which has the fixed point property, there is no similarity transformation of $C \times A$ into $C \times B$;*

(b) *for any set C such that $|C| < |C|^2$, there is no similarity transformation of $A \times C$ into $B \times C$.*

PROOF. (a) Suppose that f is a similarity transformation of $C \times A$ into $C \times B$. For each element c in C , $f(c \times A)$ is not a subset of $(d \times B)$, for some element d in B . This is so since there is no similarity transformation of A into B . Thus there are at least two elements in B , $u(c)$ and $v(c)$, $u(c) < v(c)$, such that

$$f(c \times A) \cap (u(c) \times B) \neq \emptyset \quad \text{and} \quad f(c \times A) \cap (v(c) \times B) \neq \emptyset.$$

By Lemma 2, a similarity transformation h of C into C can be found so that $h(y) \neq y$ for each element y in C . This contradicts the fact that the set C has the fixed point property. Therefore no such similarity transformation f can exist.

(b) Since $|C| < |C|^2$, it follows from Lemma 1.3 of [4] that $|C|^m < |C|^{(m+1)}$ for each positive integer m . Using this fact, it follows from an argument analogous to that given in Theorem 2 that if a similarity transformation f of $A \times C$ into $B \times C$ existed, then a similarity transformation h of A into B could be constructed. By hypothesis though, no such function h exists. Hence our result. Q.E.D.

COROLLARY 1. *If $|A| < |B|$ and C has the fixed point property, then*

$|C \times A| < |C \times B|$. If $|A| < |B|$ and $|C| < |C|2$, then $|A \times C| < |B \times C|$.

COROLLARY 2. If $|A|$ and $|B|$ are incomparable order types and C has the fixed point property, then $C \times A$ and $C \times B$ are incomparable order types. If A and B are incomparable order types and $|C| < |C|2$, then $|A \times C|$ and $|B \times C|$ are incomparable order types.

COROLLARY 3. If $|C| < |C|2$ and $|A \times C| \equiv |B \times C|$, then $|A| \equiv |B|$.

Naturally the question arises as to whether or not the conclusion of Theorem 7 remains valid if the hypotheses in (a) and (b) are reversed. The answer is in the negative as is now shown. For (a) let $|A| = 2$, $|B| = 1$, and $|C| = \omega$. Consider (b). A simply-ordered set E is said to be complete if every subset D of E has both a greatest lower bound and a least upper bound, in E . It is known that a complete set has the fixed point property [6]. Therefore the set C , where $|C| = 1 + \lambda + 1$, has the fixed point property. Let $|A| = \omega$ and $|B| = \omega^*$. Clearly there is no similarity transformation of A into B . Yet $|A \times C| = (1 + \lambda + 1)\omega \equiv \lambda \equiv (1 + \lambda + 1)\omega^* = |B \times C|$.

Using the methods of this paper one can show

THEOREM 8. If A and B each have the fixed point property, then the ordered sum $A + B$ and the product $A \times B$ each have the fixed point property.

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