THE HOMOLOGY PRODUCTS IN $K(\Pi, n)$

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1. **Introduction.** A topological space X is said to be of type $K(\Pi, n)$, for Π an abelian group and n a positive integer, if $\pi_i(X)=0$ for $i\neq n$ and $\pi_n(X)=\Pi$. The singular homology group $H(\Pi, n) = \sum_q H_q(\Pi, n)$ of such a space X is a ring. Indeed, the ring structure in $H(\Pi, n)$ has been defined in two different ways: algebraically by Eilenberg-MacLane in [**3**] and geometrically by H. Cartan (in lectures), using the Serre representation [**7**] of a suitable space $K(\Pi, n)$ as a space of loops. Eilenberg-MacLane have known that their algebraic product matched the geometric one. A proof of this assertion is now in order, especially in view of the current penetrating investigations of Cartan on $H(\Pi, n)$, which use both the geometric and the algebraic methods. Such a proof is provided below (Theorem 4) in terms of a machinery for these homology products which is analogous to the familiar "external" and "internal" descriptions of the cohomological cup products (see [**4**, §7]). We also take this occasion to record a brief proof of the identity of the standard cup products in the simplicial and the cubical singular theories.

All topological spaces X are to be arcwise connected, and are to come equipped with a fixed base point $e=e_X$; all continuous mappings are to carry the base point to the base point. In particular, as the base point of the affine q-simplex Δ_q or q-cube I^q we choose the "leading" vertex. For singular homology theory we use the notation of [**4**]; in the simplicial theory we can restrict attention to the subcomplex $S(X)=S_1(X)$ of the total singular complex of X which is spanned by those singular simplices T: $\Delta_q\to X$ which carry every vertex of Δ_q into e_X. Each such T determines, for each $i=0, \cdots, q$, a degenerate $(q+1)$-simplex D_iT: $\Delta_{q+1}\to X$ given in barycentric coordinates $\lambda_0, \cdots, \lambda_{q+1}$ by

$$(D_iT)(\lambda_0, \cdots, \lambda_{q+1}) = T(\lambda_0, \cdots, \lambda_i + \lambda_{i+1}, \cdots, \lambda_{q+1}).$$

These degenerate simplices span a subcomplex $DS(X)$, and the quotient-complex $S^N(X)=S(X)/DS(X)$ is (canonically) chain equivalent to $S(X)$. The (only) singular 0-simplex, which maps Δ_0 on e_X, will be denoted by 1_0.

Similarly, $Q(X)$ denotes the cubical singular complex of X [**2**; **7**], restricted to those singular cubes R: $I^q\to X$ which map each vertex

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of I^q to e_X. Each R determines, for each $i=1, \cdots, q+1$, a degenerate $(q+1)$-cube E_iR with

$$(E_iR)(\mu_1, \cdots, \mu_{q+1}) = R(\mu_1, \cdots, \mu_{i-1}, \mu_{i+1}, \cdots, \mu_{q+1}),$$
$$0 \leqq \mu_j \leqq 1.$$

Again $Q^N(X)$ denotes the complex Q modulo the subcomplex spanned by all E_iR. Since $\Delta_0 = I^0$ and $\Delta_1 = I^1$, the complexes $Q(X)$ and $S(X)$ are identical in dimensions 0 and 1. The complex $Q(X)$, without normalization, does not give the proper homology theory of X, even for X a point, but Q^N does. Specifically, there is a natural chain equivalence [2, Theorem V]

$$h: S(X) \to S^N(X) \to Q^N(X), \tag{1}$$

which is the composite of the factorization map $S \to S^N$ and a map $S^N \to Q^N$. This map h is the identity (modulo degenerates) in dimensions 0 and 1.

2. **The external products.** Given spaces X_1 and X_2 we identify X_1 with a subspace of the cartesian product $X_1 \times X_2$ by the map $x_1 \to (x_1, e_{X_2})$, and hence we identify $S(X_1)$—or $Q(X_1)$—with a subcomplex of $S(X_1 \times X_2)$—or $Q(X_1 \times X_2)$. The "external" products are given as follows, in terms of the usual tensor product [3, p. 68] of two complexes.

THEOREM 1. *There are natural chain transformations*

$$\begin{aligned} &\rho: Q^N(X_1) \otimes Q^N(X_2) \to Q^N(X_1 \times X_2), \\ &\sigma: S(X_1) \otimes S(X_2) \to S(X_1 \times X_2) \end{aligned} \tag{2}$$

such that, for R_i: $I^q \to X_i$ and T_i: $\Delta_q \to X_i$, $i=1, 2$,

$$\begin{aligned} \rho(R_1 \otimes 1_0) &= R_1, & \rho(1_0 \otimes R_2) &= R_2, \\ \sigma(T_1 \otimes 1_0) &= T_1, & \sigma(1_0 \otimes T_2) &= T_2. \end{aligned} \tag{3}$$

Two such transformations ρ (or two such σ) are chain homotopic. For any such ρ and σ there is a natural chain homotopy

$$H: h\sigma \simeq \rho(h \otimes h): S(X_1) \otimes S(X_2) \to Q^N(X_1 \times X_2), \tag{4}$$

where h is the simplicial to cubical chain equivalence (1).

PROOF. Given singular cubes R_1: $I^r \to X_1$ and R_2: $I^s \to X_2$, a map ρ with the specified properties has been defined by H. Cartan, who sets

$$\rho(R_1 \otimes R_2)(\mu_1, \cdots, \mu_{r+s}) = (R_1(\mu_1, \cdots, \mu_r), R_2(\mu_{r+1}, \cdots, \mu_{r+s})), \tag{5}$$

for $0 \leqq \mu_i \leqq 1$. Given singular simplices T_1: $\Delta_p \to X_1$ and T_2: $\Delta_q \to X_2$ we take a (p, q) *shuffle* (μ, ν) to be a partition of the set of integers $\{0, 1, \cdots, p+q-1\}$ into two disjoint subsets $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_q$. We may then define σ by setting $\sigma(T_1 \otimes T_2) = T_1 \bigtriangledown T_2$, where

$$(5') \qquad T_1 \bigtriangledown T_2 = \sum_{(\mu,\nu)} (-1)^{\epsilon(\mu)} (D_{\nu_q} \cdots D_{\nu_1} T_1)(D_{\mu_p} \cdots D_{\mu_1} T_2),$$

where the sum is taken over all (p, q) shuffles and where $\epsilon(\mu) = \sum (\mu_i - (i-1))$. On the right we have used a "product" of two singular simplices T_i' of X_i of the same dimension $p+q$ in $X_1 \times X_2$; this product is the simplex of $X_1 \times X_2$ defined by the formula

$$(6) \qquad (T_1' T_2')(u) = (T_1'(u), T_2'(u)), \qquad u \in \Delta_{p+q}.$$

The properties of the product $\bigtriangledown$ requisite for σ to be a chain transformation have been established [**3**, Theorem 5.2].

The analogy between (5) and (5′) may be exhibited by writing (5) in the equivalent form

$$\rho(R_1 \otimes R_2) = (E_{r+s} \cdots E_{r+1} R_1)(E_r \cdots E_1 R_2),$$

and observing that the various terms in (5′) correspond exactly to the terms in the standard simplicial subdivision of $\Delta_r \times \Delta_s$ [**3**, p. 63].

To construct the homotopy H of (4), observe first that the 1-cells of $S(X_1) \otimes S(X_2)$ are necessarily of the form $T_1 \otimes 1_0$ or $1_0 \otimes T_2$, for T_1 or T_2 of dimension 1. Hence, by (3), $h\sigma = \rho(h \otimes h)$ in dimensions 0 or 1, so that we may take H to be zero on these dimensions. Suppose by induction that a natural H has been defined in all dimensions less than m, for $m = 2$ or greater. To define $H(T_1 \otimes T_2)$ for T_1, T_2 of dimensions p, q with $p+q = m$, observe that we may write $T_1 = T_{1*} b_p$, where T_{1*}: $S(\Delta_p) \to S(X_1)$ is the chain transformation induced by T_1, and b_p: $\Delta_p \to \Delta_p$ is the identity map. Hence

$$T_1 \otimes T_2 = (T_{1*} \otimes T_{2*})(b_p \otimes b_q) \in S(X_1) \otimes S(X_2).$$

If H is to be natural, then

$$(7) \qquad H(T_1 \otimes T_2) = (T_1 \times T_2)_* H(b_p \otimes b_q) \in Q^N(X_1 \times X_2),$$

where $T_1 \times T_2$: $\Delta_p \times \Delta_q \to X_1 \times X_2$ is the cartesian product of the maps T_1, T_2. It thus suffices to define $H(b_p \otimes b_q)$. To this end consider the chain

$$c = [-H\partial + h\sigma - \rho(h \otimes h)](b_p \otimes b_q)$$

of dimension $m = p+q$ in $Q^N(\Delta_p \times \Delta_q)$. By routine calculation $\partial c = 0$; since $\Delta_p \times \Delta_q$ is contractible, c is the boundary of some chain a. Then $H(b_p \otimes b_q) = c$ and (7) define H; the form of a and the naturality of

H, h, σ, and ρ show that $\partial H + H\partial = h\sigma - \rho(h \otimes h)$, while the form of (7) proves H natural.

This argument is an instance of the method of acyclic models, exactly as described in general in [**3**]. The same method will prove that any two chain transformations ρ (or any two σ) are *naturally* chain homotopic. One may also use this method to prove the existence of ρ and σ without recourse to the explicit formulas (5) and (5′).

REMARK. The explicit map σ constructed in (5′) has the additional property [**3**, Lemma 5.3] that

$$DS(X_1) \nabla S(X_2) \cup S(X_1) \nabla DS(X_2) \subset DS(X_1 \times X_2);$$

hence this σ induces a natural chain transformation modulo degenerates,

$$S^N(X_1) \otimes S^N(X_2) \to S^N(X_1 \times X_2). \tag{8}$$

The theorem as of σ may be sharpened by requiring that σ so induce such a map (8), and it may then be asserted that the induced maps (8), for two such σ, are chain homotopic.

3. ***H*-complexes.** X is an H-space [**7**, p. 474] if there is a function $(x, y) \to x \vee y$ which is a continuous map $\vee: X \times X \to X$ with $e \vee e = e$, and such that the maps $x \to x \vee e$ and $x \to e \vee x$ are both homotopic to the identity map $X \to X$ by homotopies leaving e fixed.

All chain complexes K to be considered have no chains (except 0) of negative dimensions, and in addition have the additive group J of integers as the group of chains of dimension zero, and have 0 as the group of boundaries of dimension 0. (Clearly $S(X)$, $S^N(X)$, and $Q^N(X)$ are such complexes.) Such a complex K is automatically augmentable, and $H_0(K)$ is isomorphic to J. We use this isomorphism to identify $H_0(K)$ with J.

A *homology product* in K is a homomorphism M: $H(K) \otimes H(K) \to H(K)$ such that $M(1 \otimes u) = u = M(u \otimes 1)$ and such that the dimension of $M(u \otimes w)$ is the sum of the dimensions of u and w, for u, $w \in H(K)$. Such a product M is by definition bilinear but need not be associative.

An *H-complex* K is a complex as above together with a chain transformation P: $K \otimes K \to K$ such that $P(1 \otimes 1) = 1$, while both chain transformations $a \to P(a \otimes 1)$ and $a \to P(1 \otimes a)$ of K into K are chain homotopic to the identity map on K. Each such P gives a homology product as the composite

$$H(K) \otimes H(K) \xrightarrow{\alpha} H(K \otimes K) \xrightarrow{P_*} H(K),$$

where α is the familiar map from the Künneth relations, defined for cycles y and z in the homology classes y' and z' by $\alpha(y'\otimes z')=(y\otimes z)'$.

If K and K' are H-complexes, an *H-map* f: $K\to K'$ is a chain transformation f: $K\to K'$ for which there exists a chain homotopy

$$P'(f \otimes f) \simeq fP : K \otimes K \to K'. \tag{9}$$

It follows at once that the induced map $f_*: H(K)\to H(K')$ is a homomorphism for the homology products.

Now let X be an H-space. The multiplication $\vee$ in X and the external products ρ and σ of Theorem 1 then induce chain transformations

$$S(X) \otimes S(X) \xrightarrow{\sigma} S(X \times X) \xrightarrow{\vee_*} S(X), \tag{10}$$

$$Q^N(X) \otimes Q^N(X) \xrightarrow{\rho} Q^N(X \times X) \xrightarrow{\vee_*} Q^N(X). \tag{10'}$$

THEOREM 2. *If X is an H-space, the chain transformations $P=\vee_*\sigma$ or $P=\vee_*\rho$ of* (10) *or* (10′) *make $S(X)$ and $Q^N(X)$ H-complexes. The induced homology products in $H(S(X))$ and $H(Q^N(X))$ are independent of the choice of the maps σ and ρ, subject to the conditions of Theorem* 1.

PROOF. That $P(1_0\otimes 1_0)=1_0$ is clear in either case. To study $T\to P(T\otimes 1_0)$, observe first by (3) that $P(T\otimes 1_0)=\vee_*\sigma(T\otimes 1_0)$ $=\vee_* T$, where the last T is regarded as a singular simplex T: Δ_q $\to X\times e\to X\times X$ in $X\times X$. Hence, if T has dimension q and d is a point of Δ_q,

$$P(T \otimes 1_0)(d) = T(d) \vee e.$$

On the other hand, let ϕ: $X\to X$ for the moment denote the map $\phi(x)=x\vee e$; then

$$(\phi T)(d) = T(d) \vee e.$$

Therefore $\phi T=P(T\otimes 1_0)$ and the chain transformation $T\to P(T\otimes 1_0)$ is exactly the chain transformation ϕ_*: $S(X)\to S(X)$. By the definition of an H-space, ϕ_* is chain homotopic to the identity, which shows that P makes $S(X)$ an H-complex. A similar argument applies to Q^N.

If σ' is a second choice of the map σ, then, by Theorem 1, $\sigma\simeq\sigma'$, hence $P=\vee_*\sigma\simeq\vee_*\sigma'=P'$. This means that the identity map i: $K\to K$ is an H-map for P and P', and hence that the homology products are independent of the choice of σ.

Note, by the remark above, that Theorem 2 is also valid for the normalized complex S^N.

THEOREM 3. *If X is an H-space, the chain equivalence h: $S(X) \to Q^N(X)$ of* (1) *induces an isomorphism for the homology products defined in the simplicial and cubical complexes by Theorem* 2.

PROOF. It suffices to prove h an H-map. By (9), this amounts to proving commutativity up to a homotopy in the diagram formed from (10) and (10′) by adding the vertical maps $h \otimes h$, h, and h. The right-hand square is exactly commutative, since h is natural; the left-hand square is commutative up to a homotopy by (4) of Theorem 1.

These results may be summarized in the statement that the singular homology group of an H-space has a multiplicative structure uniquely determined by the multiplication given in the H-space.

4. **The space of loops.** Let Π be an abelian group. By [6] or [10] we may construct a space Y of type $K(\Pi, n+1)$. Take X to be the space of loops in Y at the base point e_Y; in other words, X is the space of all maps x: $(I, \dot{I}) \to (Y, e_Y)$, with the compact-open topology, and with base point $e = e_X$ the degenerate loop with $e(\mu) = e_Y$, $0 \leqq \mu \leqq 1$. Then Serre [7] has shown that X is a space of type $K(\Pi, n)$ and is an H-space relative to the (usual) composition of loops. The results of the previous section then introduce a homology product in $H(\Pi, n) = H(X)$. This is the product used by Cartan for the cubical theory.

Now consider any space X of type $K(\Pi, n)$ which is also an H-space (X need not be the space of loops in Y, as above). The following simple "geometric" addition theorem is essentially well known (and was suggested to me in this connection by H. Cartan).

LEMMA. *For X an H-space and maps f, g: $(I^n, \dot{I}^n) \to (X, e)$ let $\{f\}$, $\{g\}$ denote the corresponding elements of $\pi_n(X)$, and define $f \vee g$: $(I^n, \dot{I}^n) \to (X, e)$ by setting $(f \vee g)(d) = f(d) \vee g(d)$, for $d \in I^n$. Then*

$$\{f \vee g\} = \{f\} + \{g\} \qquad \text{in } \pi_n(X). \tag{11}$$

PROOF. Let $f+g$ denote the usual composition of the maps f and g, as used to define the addition in $\pi_n(X)$, and let u: $(I^n, \dot{I}^n) \to (X, e)$ be the map taking all of I^n into $e = e_X$. We then have homotopies (rel $\dot{I}^n$) $f+u \simeq f$ and $g \simeq u+g$; the H-composite of these homotopies yields a homotopy

$$f \vee g \simeq (f+u) \vee (u+g) = (f \vee u) + (u \vee g).$$

The homotopies given in the definition of an H-space now show that $f \vee u \simeq f$, $u \vee g \simeq g$ (rel $\dot{I}^n$), and hence that $f \vee g \simeq f+g$, as required for (11).

We actually need this lemma with I^n, $\dot{I}^n$ replaced by Δ_n, $\dot{\Delta}_n$. The usual transition from the representation of $\pi_n(X)$ by cubes to that by simplices at once proves the lemma in this form, mutatis mutandis.

The algebraic products in $H(\Pi, n)$ are introduced in [3], as follows. $H(\Pi, n)$ is the homology group of a "standard" complete semi-simplicial complex $K(\Pi, n)$; the q-simplices u of $K(\Pi, n)$ are the cocycles $u \in Z^n(\Delta_q; \Pi)$. This complex $K(\Pi, n)$ is an H-complex under the composite map

$$(12) \qquad K(\Pi, n) \otimes K(\Pi, n) \xrightarrow{\nabla} K(\Pi, n) \times K(\Pi, n) \xrightarrow{p} K(\Pi, n).$$

Here ∇ is the map given by the formula (5′), using the degeneracy operators D_i represent in the complete semi-simplicial (or FD) complex $K(\Pi, n)$. The complex $K(\Pi, n) \times K(\Pi, n)$ has as its q-simplices the formal products $u \times v$, for $u, v \in Z^n(\Delta_q; \Pi)$. The second map p is that induced by the group operation in Π; specifically $p(u \times v)$ is that n-cocycle in Δ_q given as the sum of u and v; that is, $p(u \times v)(\tau) = u(\tau) + v(\tau)$ for τ an n-face of Δ_q.

To compare the algebraic and geometric definitions, first replace $S(X)$ by $S_n(X)$, defined as that subcomplex of $S(X)$ which is spanned by all those singular simplices $T: \Delta_q \to X$ which map the $(n-1)$-skeleton of Δ_q to e_X. Since $\pi_1(X) = \cdots = \pi_{n-1}(X) = 0$, the identity injection $i_n: S_n(X) \to S(X)$ is a chain equivalence. Using σ exactly as in Theorem 2, $S_n(X)$ becomes an H-complex.

Since X is of type $K(\Pi, n)$ there is a chain equivalence $\kappa: S_n(X) \to K(\Pi, n)$ defined as follows. For each $T: \Delta_q \to X$ let $\kappa T \in Z^n(\Delta_q; \Pi)$ be that cocycle whose value on any n-face τ of Δ_q is the element of $\pi_n(X)$ represented by the map T cut down to that n-face; in symbols,

$$(\kappa T)(\tau) = \{T \mid \tau\}.$$

THEOREM 4. *If X is an H-space of type $\kappa(\Pi, n)$, then the chain equivalences*

$$S(X) \xrightarrow{i_n} S_n(X) \xrightarrow{\kappa} K(\Pi, n)$$

are H-maps and hence induce isomorphism on the homology products introduced above in each of these complexes.

PROOF. Regard $S(X)$ and $S_n(X)$ as H-complexes under the standard map σ given by (5′). It is immediate that i_n is an H-map (and it follows formally that any homotopy inverse of i_n is also an H-map). To show that κ is an H-map we shall establish commutativity (exactly!) in the diagram

$$\begin{array}{ccccc} S_n(X) \otimes S_n(X) & \xrightarrow{\nabla} & S_n(X \times X) & \xrightarrow{\vee_*} & S_n(X) \\ \downarrow{\scriptstyle \kappa \otimes \kappa} & & \downarrow{\scriptstyle \kappa \times \kappa} & & \downarrow{\scriptstyle \kappa} \\ K(\Pi, n) \otimes K(\Pi, n) & \xrightarrow{\nabla} & K(\Pi, n) \times K(\Pi, n) & \xrightarrow{p} & K(\Pi, n). \end{array}$$

The left-hand square is commutative because ∇ is a natural map (for FD-complexes, in the sense of [**3**]). To show the right-hand square commutative, write any q-simplex of $X \times X$ in the form $T = T_1 \times T_2$, where $T_i \colon \Delta_q \to X$. Its image by either path in the diagram is a q-simplex of $K(\Pi, n)$, and thus an n-cocycle in $Z^n(\Delta_q; \Pi)$. We calculate the value of the cocycle on an n-face τ of Δ_q, as follows.

$$\begin{aligned} [p(\kappa \times \kappa)T]\tau &= [p(\kappa T_1 \times \kappa T_2)]\tau = (\kappa T_1)\tau + (\kappa T_2)\tau \\ &= \{T_1 \mid \tau\} + \{T_2 \mid \tau\}, \\ [\kappa \vee_*(T_1 \times T_2)]\tau &= [\kappa(T_1 \times T_2)]\tau = \{(T_1 \vee T_2) \mid \tau\}. \end{aligned}$$

By the lemma (in simplicial form) the two results agree.

COROLLARY. *The homology products in an H-space X of type $\mathcal{K}(\Pi, n)$ are independent of the choice of X and are associative and skew-commutative.*

PROOF. The independence is clear, since $K(\Pi, n)$ does not depend on the choice of X; on the other hand the chain transformation $p\nabla$ used in (12) to make $K(\Pi, n)$ an H-complex is known [**3**, Theorem 6.1] to be associative and skew-commutative; hence the second result.

This corollary can also be obtained by geometrical arguments on the loop spaces, using for the independence argument suitable techniques of combinatorial homotopy [**9**].

The complex $K(\Pi, n)$ is an algebraic replica of the minimal complex of a space $\mathcal{K}(\Pi, n)$ in the simplicial theory. H. Cartan, in lectures, has constructed an analogous replica for the cubical theory. The product operation in this case is simpler—like (5) rather than (5′)—but it is not skew-commutative, and it appears to offer no analogue for the "half products" used in [**3**] to prove the equivalence of $K(\Pi, n)$ to n bar constructions on the integral group ring of Π.

5. **Cubical and simplicial cup products.** Serre [**7**] uses a cup product for the cubical singular cohomology theory; we record here a proof that this cup product agrees (as is intuitively clear) with the usual Čech-Alexander cup product in the simplicial singular cohomology theory. By known techniques [**4**, §7] a cup product in the cohomology of a chain complex K can be obtained from a chain

transformation $e\colon K \to K\otimes K$. The Čech-Alexander product in $S(X)$ is obtained from the composite

$$S(X) \xrightarrow{d_*} S(X\times X) \xrightarrow{f} S(X)\otimes S(X),$$

where d is the diagonal map $d\colon X\to X\times X$ with $d(x)=(x, x)$, and where f can be defined more generally as a chain transformation $f\colon S(X_1\times X_2) \to S(X_1)\otimes S(X_2)$ according to the formula

$$f(T_1\times T_2) = \sum_{i=0}^{q} (F_q\cdots F_{i+1}T_1)\otimes (F_{i-1}\cdots F_0 T_2), \tag{13}$$

for $T_i\colon \Delta_q\to X_i$ and F_i the usual face operators. The cup products in $Q^N(X)$ may be obtained in exactly parallel form, using a chain transformation $g\colon Q^N(X_1\times X_2)\to Q^N(X_1)\otimes Q^N(X_2)$ defined for n-cubes $R_i\colon I^n\to X_i$ by means of the cubical face operators A_i and B_i of [2] as

$$g(R_1\times R_2) = \sum_{(\mu,\nu)} (-1)^{\rho(\mu,\nu)}(A_{\mu_1}\cdots A_{\mu_p}R_1)\otimes(B_{\nu_1}\cdots B_{\nu_q}R_2), \tag{13'}$$

where the sum is taken over all partitions of the set $\{1, \cdots, n\}$ into two disjoint sets $\mu_1<\cdots<\mu_p$ and $\nu_1<\cdots<\nu_q$, with $p+q=n$, and where $\rho(\mu, \nu)$ is the number of pairs of indices i, j with $\mu_i<\nu_j$. With the notational change for the faces $A_i\to\lambda_i^0$, $B_i\to\lambda_i^1$, it is clear that the formula (13′) corresponds exactly to the cup-products as introduced by Serre [7, p. 441]. The technique of acyclic models will establish the agreement of these two formulas—without appeal to the explicit formulas!

THEOREM 5. *There exist natural chain transformations*

$$f\colon S(X_1\times X_2)\to S(X_1)\otimes S(X_2),$$
$$g\colon Q^N(X_1\times X_2)\to Q^N(X_1)\otimes Q^N(X_2)$$

which are the identity in dimension 0. *Any two such transformations f are chain homotopic, and likewise for any two g.*

PROOF. (Recall that in dimension 0 each of the complexes is simply the group of integers.) The map f is precisely one of the maps of the Eilenberg-Zilber Theorem and is constructed by the method of acyclic models in [5]. The same method proves the existence of a g and the homotopy of any two such g's.

The maps f and g of (13) and (13′) are maps of the type specified in this theorem.

THEOREM 6. *For maps f and g as in Theorem* 5, *and for h as in* (1), *the diagram*

$$\begin{array}{ccccc} S(X) & \xrightarrow{d_*} & S(X \times X) & \xrightarrow{f} & S(X) \otimes S(X) \\ \downarrow h & & \downarrow h & & \downarrow h \otimes h \\ Q^N(X) & \xrightarrow{d_*} & Q^N(X \times X) & \xrightarrow{g} & Q^N(X) \otimes Q^N(X) \end{array}$$

is commutative up to a homotopy.

Since the horizontal maps are those used to define the cup products, while the vertical maps are those used to compare the simplicial and cubical theories, it follows readily that this commutativity establishes the agreement of the cup products on homology classes in the two homology theories.

PROOF. The left-hand square is commutative, since h is natural. Replace the right-hand square by the more general

$$\begin{tikzcd} S(X_1 \times X_2) \arrow[r, "f"] \arrow[d, shift right, "h"'] & S(X_1) \otimes S(X_2) \arrow[d, "h \otimes h"] \\ Q^N(X_1 \otimes X_2) \arrow[r, "g"] \arrow[u, shift right, "k"'] & Q^N(X_1) \otimes Q^N(X_2) \end{tikzcd}$$

where k is a homotopy inverse of h. Then $(h \otimes h)fk$ is a map with the same formal properties as g, and in particular is the identity in dimension 0. Hence, by Theorem 5,

$$(h \otimes h)fk \simeq g$$

and therefore, since $kh \simeq 1$, $(h \otimes h)f \simeq gh$, which gives the result desired.

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