ON A LOCALLY COMPACT GROUP ACTING ON A MANIFOLD

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Introduction. Let G be a locally compact group, admissibly topologized [**5**], operating as a transitive transformation group on a manifold M. Using the notation of [**1**], let G_x be the subgroup of G which leaves the point x fixed. Montgomery [**1**] has proved the following result.

THEOREM (MONTGOMERY). *If G is a connected Lie group which acts transitively on a compact manifold M, and if G_x is connected, then G contains a compact subgroup which acts transitively on M.*

The purpose of this note is to show that the theorem is true when G is any locally compact group satisfying the second axiom of countability. It will follow, by [**4**], that the corollaries of [**1**] are true for the more general case.

1. A lemma. Arens [**5**] has proved that M is homeomorphic to G/G_x. The author [**2**; **3**] has proved that G is a fibre bundle over G/G_x.

LEMMA. *Let G be a connected locally compact group satisfying the second axiom of countability and acting transitively on a manifold M. Then there exists a Lie group G^* acting transitively on M such that G^* is a factor group of G by a compact normal subgroup of G_x.*

PROOF. Let N be a compact normal subgroup of G such that G/N is a Lie group [**6**], and $N^* = N \cap G_x$. We shall show that $G/N^* = G^*$ is a Lie group which acts transitively on M. As mentioned above, the author [**3**] has shown that G/N^* is a fibre bundle over G/G_x with fibre G_x/N^*, and hence G/N^* is locally homeomorphic to $(G_x/N^*) \times K$ where K is homeomorphic to a euclidean space. Hence, G/N^* is a manifold. Now G/G_x is homeomorphic and isomorphic to $(G/N)/(G_x/N^*)$ so that by a proper choice of N, dim $(N/N^*) = 0$. Since G/N^* is a fibre bundle over G/N, G/N^* is locally homeomorphic with $(N/N^*) \times L$ where L is a euclidean space. Hence, N/N^* is discrete. This implies that a subgroup of N^* (which we may assume to be N^* itself) has the property that G/N^* is a Lie group. Define, for $g^* \in G^*$, $g^* = gN^*$, $y \in M$,

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$$g^*(y) = gN(y).$$

It is not difficult to show that G^* forms a transformation group on M with this definition. Since $N^* \subset G_x$, G^* is transitive.

2. **The theorem.** Under the hypothesis of the lemma, and if G_x is moreover connected, then Montgomery's theorem implies that G^* contains a compact subgroup H^* which acts transitively on M for M compact. If p is the natural projection $G \to G^*$, then $p^{-1}(H^*) = H$ is compact and acts transitively on M. Hence, we have proved the theorem as stated below.

THEOREM. *Let G be a connected locally compact group satisfying the second axiom of countability which acts transitively on a compact manifold. If G_x is connected, then G contains a compact subgroup which acts transitively on M.*

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