and for each particular f the measure may be so chosen that (moreover)

$$\int_B \log |f(x)| \, m_s(dx) \geqq \log |f(s)|. \tag{7.13}$$

Naturally, if for some reason there is for some s only one measure satisfying 7.12, then 7.13 holds for that measure.

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FAMILIES OF CURVES

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Amasa Forrester in [1] proved the following theorem of a mixed Euclidean and topological character. If ϕ is a continuous map without fixed points on the Euclidean n-sphere such that ϕ^2 is the identity, then the chords $P\phi(P)$ for all points P of the sphere completely fill the interior of this sphere.

The object of this note is to generalize this theorem to a purely topological statement.

First we recall the definition of retract. If $B \subset A$ are two spaces, then B is a retract of A if there is $r: A \to B$ which leaves fixed all points of B. (If X and Y are spaces the symbol $f: X \to Y$ shall denote a continuous map from X to Y.)

Let I denote the unit interval. If $F: B \times I \to A$ and $t \in I$, define $F_t: B \to A$ by $F_t(b) = F(B, t)$ for all $b \in B$.

Observation. If $F: B \times I \to A$ and if B is a retract of A by the map r and if $p, q \in I$, then rF_p is homotopic to rF_q.

In fact such a homotopy is provided by $G: B \times I \to B$ defined by

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$G(b, t) = rF(b, p+(q-p)t)$. Clearly $G_0 = rF_p$ and $G_1 = rF_q$.

Now let E^{n+1} be the topological $n+1$ dimensional cell and S^n its boundary (an n dimensional topological sphere). Now, for any point $P \in E^{n+1} - S^n$, S^n is a retract of $E^{n+1} - \{P\}$ by the map

$$r: E^{n+1} - \{P\} \to S^n$$

defined by carrying over the central projection of the Euclidean cell by a homeomorphism. This fact and the observation yield:

PROPOSITION 1. *If f_i: $S^n \to S^n$, $i=0, 1$, are not homotopic and if F: $S^n \times I \to E^{n+1}$ satisfies $F(P, i) = f_i(P)$, $i=0, 1$, all $P \in S^n$, then F is onto E^{n+1}.*

PROPOSITION 2 (*generalization of Forrester's theorem*). *Let ϕ: $S^n \to S^n$ be of period $p \neq 1$. Let F: $S^n \times I \to E^{n+1}$ satisfy* (a) $F(P, 0) = P$ *and* (b) $F(P, 1) = F(\phi(P), 1)$. *Then F is onto E^{n+1}.*

PROOF. Observe first that it is sufficient to prove this for p prime. For if p were not prime and q is a prime dividing p, then the hypothesis of Proposition 2 is satisfied with ϕ replaced by $\phi^{p/q}$ and the latter is of prime period. In the following proof therefore p is taken to be prime.

Assume on the contrary that there is a point $Q \in E^{n+1} - F(S^n \times I)$. By (a), $Q \notin S^n$ and by a previous remark there is a retraction r: $E^{n+1} \to \{Q\} \to S^n$. Regarding F as a map into $E^{n+1} - Q$ we would have rF_0 homotopic to rF_1. Now rF_0 is the identity map of S^n (hence of degree 1) while rF_1 has the property that $(rF_1)\phi = rF_1$ on account of condition (b).

To conclude the proof it shall be shown that any map g: $S^n \to S^n$ satisfying $g\phi = g$ has a degree divisible by p.

By [2] there is a cycle of the form $c + \phi(c) + \phi^2(c) + \cdots + \phi^{p-1}(c)$ in a generator of $H^n(S^n, J_p)$. Calling this cycle z we have $g(z) = pc = 0$ (mod p). Thus the degree of g is divisible by p. This concludes the proof of Proposition 2.

Forrester's family of straight lines may be described by

$$F: S^n \times I \to E^{n+1}$$

where $F(P, t)$ is the point Q on the line segment joining P to $\phi(P)$ such that $PQ/P\phi(P) = t/2$.

If the notion of homotopy is translated into the language of a continuous family of curves then Proposition 2 becomes:

PROPOSITION 2′. *If* (1) *ϕ: $S^n \to S^n$ satisfies the condition stated in Proposition* 2 *and* (2) *from each point P of S^n there begins one curve of*

E^{n+1} so that the curves beginning at P and $\phi(P)$ have the same terminal point and (3) *the parametrization of these curves depend continuously on P, then this family of curves fills E^{n+1}.*

PROPOSITION 3. *Let R^n refer to n dimensional Euclidean space. If for each direction in R^n there is given in a continuous manner precisely one straight line with that direction, then this family of lines fills R^n.*

PROOF. Compactify R^n to E^n by adding two points at infinity for each direction in R^n. Then apply Proposition 2 or 2′ with $p=2$.

PROPOSITION 4. *Let A be a compact subset of R^n. A necessary and sufficient condition that A be a convex set with the property that each support plane has precisely one contact point is that there exists a continuous choice function on the set of $n-1$ dimensional planes, meeting A, with values in A. Moreover any such function is onto A.*

PROOF. Let A be a convex subset of R^n with the property that each plane of support has one point of contact. Assign to each cross-section its centroid. By Proposition 2 with $p=2$ it is easy to show that this function is onto A (compare to p. 13 of [**3**]).

The proof of sufficiency is left to the reader. In a subsequent paper the intersection properties of families of curves will be considered.

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