A NOTE ON THE REPRESENTATIONS OF NILPOTENT LIE ALGEBRAS¹

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1. Introduction. H. Zassenhaus has determined² the irreducible representations of a nilpotent Lie algebra $\mathfrak{L}$ over an algebraically closed field K of characteristic $p > 0$, there being exactly one for each ordered set of characteristic values corresponding to the elements of a regular basis of $\mathfrak{L}$. The question as to whether the one-to-one correspondence persists between the irreducible representations of $\mathfrak{L}$ and ordered sets of irreducible polynomials corresponding to the elements of a regular basis of $\mathfrak{L}$ when K is replaced by an arbitrary field of characteristic $p > 0$ is answered affirmatively in this note. The existence of the representation belonging to a prescribed set of irreducible polynomials is proved as an application of two general results (Propositions 4 and 5 below), the first of which is a generalization of Zassenhaus' theorem³ on the existence of irreducible $\mathfrak{L}$ - σ modules containing a given irreducible $\mathfrak{L}$ -module. Rather than dealing directly with the representation spaces, our method exploits the ideal theoretic properties of the universal associative algebra⁴ of $\mathfrak{L}$.

Our discussion begins with some results concerning the indecomposable representations of a nilpotent Lie algebra over an arbitrary field, and we obtain as a main result the theorem that any two irreducible constituents of such a representation are equivalent. A key step in the proof is the observation that the minimum polynomials of the representation transformations are powers of irreducible polynomials. The treatment of this point is due to Professor N. Jacobson, and was presented by him in his course at Yale in 1950–51.

The subjects considered here were brought to our attention in conversation with Professor Jacobson, and we record here our appreciation to him for many helpful suggestions concerning these matters.

2. Indecomposable representations of nilpotent Lie algebras. We

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² See [9] and [10] in the list of References at the end of the paper.

³ See [9, p. 74].

⁴ A definition of this algebra, and the results concerning it that are needed in this paper, can be found in [1] and in [8].

begin with some preliminary results. Let K be an arbitrary field. If f is a polynomial in $K[X]$ of degree k , $f(X) = \sum_{i=0}^k \beta_i X^i$, then for each j , $0 \leq j \leq k$, we define $f_j(X) = \sum_{i=0}^j C_{i,j} \beta_i X^{i-j}$. The mapping $f \rightarrow f_j$ is a l.t.⁵ in $K[X]$. We can then establish Leibniz's rule

$$(1) \quad (fg)_i = \sum_{i=0}^j f_i g_{j-i},$$

by observing first, by a direct calculation, that it is valid for $f(X) = X^r$, $g(X) = X^s$, and that it holds in $K[X]$ by linearity.

LEMMA 1. *If f and g are polynomials in $K[X]$ such that $f^{j+1} | g$, then $f | g_i$, $i=0, 1, \dots, j$.*

PROOF. We use induction on j , noting that the result is clear if $j=0$. Assume that it holds for $j-1$, and let $f^{j+1} | g$. Then we can write $g = hk$, where $f^j | h$ and $f | k$. By induction, $f | h_0, \dots, h_{j-1}$. Then by (1), $g_j = \sum_{i=0}^j h_i k_{j-i}$, and $f | g_j$ as required.

Before stating the next lemma, we list a number of formulas which we shall need. Let $\mathfrak{R}$ be an associative ring, and let $a, d \in \mathfrak{R}$. Then

$$(2) \quad ad^k = d^k a + C_{k,1} d^{k-1} a' + C_{k,2} d^{k-2} a'' + \dots + a^{(k)},$$

$k=1, 2, \dots$, where $a' = [a, d]$, and $a^{(i)}$ is defined recursively by $a^{(i)} = [a^{(i-1)}, d]$. If $\mathfrak{R}$ has characteristic $p > 0$, then we obtain from (2)

$$(3) \quad ad^p = d^p a + a^{(p)}, \text{ and } ad^{pk} = d^{pk} a + a^{(pk)}, \quad k=1, 2, \dots$$

On the other hand, if $\mathfrak{R}$ is an algebra over K , then another application of (2) yields

$$(4) \quad af(d) = f(d)a + f_1(d)a' + f_2(d)a'' + \dots + f_k(d)a^{(k)}, \quad f \in K[X].$$

Now let A and D be l.t. in a finite-dimensional vector space $\mathfrak{M}$ over K . For each polynomial $f \in K[X]$, we define $\mathfrak{M}_{f,D}$ to be the subspace of $\mathfrak{M}$ consisting of those vectors y that are annihilated by some power of $f(D)$. Then we obtain the following result.

LEMMA 2. *If A and D are l.t. such that $A^{(n)} = [[\dots [AD]D] \dots D] = 0$ (n D 's) for some n , then $\mathfrak{M}_{f,D} A \subseteq \mathfrak{M}_{f,D}$, for every $f \in K[X]$.*

PROOF. Let $y \in \mathfrak{M}_{f,D}$, and suppose that $y(f(D))^m = 0$. If we set $g = f^m$, $h = g^n = f^{mn}$, then by (4) we have

$$(5) \quad Ah(D) = h(D)A + h_1(D)A' + \dots + h_{n-1}(D)A^{(n-1)}.$$

Since $g^n | h$, we have by Lemma 1, $g | h_i$, $i=0, 1, \dots, n-1$, and by

⁵ We shall use the abbreviation l.t. for linear transformation.

(5), $yAh(D)=0$, proving that $yA \in \mathfrak{M}_{f,D}$.

PROPOSITION 1. *Let $\mathfrak{L}$ be an indecomposable Lie algebra of l.t. in a finite-dimensional vector space $\mathfrak{M}$ over K , and let D be a l.t. such that $[\mathfrak{L}, D] \in \mathfrak{L}$. If the derivation $A \rightarrow [A, D]$ in $\mathfrak{L}$ is nilpotent, then the minimum polynomial of D is a power of an irreducible polynomial.*

PROOF. Let f be the minimum polynomial of D , and let $f = h_1^{e_1} \cdots h_r^{e_r}$ be the prime factorization of f in $K[X]$, where the h_i are distinct irreducible polynomials. Suppose that $r > 1$; then we have $\mathfrak{M} = \mathfrak{M}_{h_1,D} \oplus \cdots \oplus \mathfrak{M}_{h_r,D}$, where the $\mathfrak{M}_{h_i,D} \neq (0)$. On the other hand, for each $A \in \mathfrak{L}$, $A^{(n)} = [[A, D] \cdots D] = 0$ for some n , and Lemma 2 implies that the subspaces $\mathfrak{M}_{h_i,D}$ are invariant relative to $\mathfrak{L}$, contradicting our hypothesis that $\mathfrak{L}$ is indecomposable.

COROLLARY. *Let $\mathfrak{L}$ be an indecomposable nilpotent Lie algebra of l.t. in a finite-dimensional vector space. Then the minimum polynomial of every $A \in \mathfrak{L}$ is a power of an irreducible polynomial.*

PROPOSITION 2. *Let $x \rightarrow U_x$ be a representation of finite degree of a nilpotent Lie algebra $\mathfrak{L}$ over an infinite field K . If⁶ $\det U_x = 0$ for all $x \in \mathfrak{L}$, then there exists a nonzero vector v in the representation space such that $vU_x = 0$ for every $x \in \mathfrak{L}$.*

PROOF. This result has been proved by Chevalley [3] for fields of characteristic zero; we shall indicate how his proof can be modified so that it holds for an infinite field K of characteristic $p > 0$. Let $U(\mathfrak{L})$ be the set of l.t. U_x , $x \in \mathfrak{L}$, and let $\mathfrak{M}$ be the representation space of U . The proof is carried out by induction on the dimension of $\mathfrak{M}$, the result being clear if the dimension is one. First we suppose that U is an irreducible representation. By (3) there exists a k independent of x such that the p^k th power of each element of $U(\mathfrak{L})$ is in the center of the enveloping algebra of $U(\mathfrak{L})$, and by Schur's Lemma, if $U_x^{p^k} \neq 0$, then $U_x^{p^k}$ is an automorphism of $\mathfrak{M}$, and U_x cannot be singular. Thus U_x is nilpotent for every $x \in \mathfrak{L}$. By a theorem of Jacobson [7] the enveloping algebra $\mathfrak{A}$ of $U(\mathfrak{L})$ is a nilpotent algebra, and it follows that $\mathfrak{M}\mathfrak{A} = (0)$. Therefore $\mathfrak{M}U(\mathfrak{L}) = (0)$.

The remainder of the proof is identical with the characteristic zero case, and for the sake of completeness, we shall include it. If U is reducible, then we select a maximal submodule $\mathfrak{P}$ of $\mathfrak{M}$; by the first part of the proof we may assume that $\mathfrak{P} \neq (0)$. Let U' be the induced representation in $\mathfrak{M} - \mathfrak{P}$, and let U'' be the restriction of U to $\mathfrak{P}$. Then we have $\det U_x = \det U'_x \cdot \det U''_x = 0$ for all $x \in \mathfrak{L}$. The func-

⁶ By $\det U_x$ we mean the determinant of a matrix belonging to U_x .

tions $U'_x \rightarrow \det U'_x$ and $U''_x \rightarrow \det U''_x$ are polynomial functions on the subspaces of the algebras of l.t. of $\mathfrak{M} - \mathfrak{P}$ and $\mathfrak{P}$ generated by the sets $U'(\mathfrak{L})$ and $U''(\mathfrak{L})$ respectively. Since K is an infinite field, one factor must vanish identically [2, p. 27]. If $\det U''_x = 0$ for all $x \in \mathfrak{L}$, then our induction hypothesis yields the result, so we may suppose that for some $a \in \mathfrak{L}$, $\det U''_a \neq 0$, and that $\det U'_x = 0$, $x \in \mathfrak{L}$. By the first part of the proof we have $U'(\mathfrak{L}) = (0)$, so that $\mathfrak{M}U(\mathfrak{L}) \subseteq \mathfrak{P}$. Now since $\det U_a = 0$, for some $y \in \mathfrak{P}$, $yU_a = 0$. Let $\mathfrak{Y} = yU(\mathfrak{L})$. Then $\mathfrak{Y} \subseteq \mathfrak{P}$, and for each $x \in \mathfrak{L}$, $yU_x U_a = yU_{[x,a]} + yU_a U_x = yU_{[x,a]}$ since $yU_a = 0$. By induction we have $yU_x U_a^k = yU_{[\dots[x,a]\dots a]} = 0$ (k a 's) for sufficiently large k . If some $yU_x \neq 0$, then we can find k such that $z = yU_x U_a^{k-1} \neq 0$, and $zU_a = 0$. Since $z \in \mathfrak{P}$, this contradicts our assumption that U''_a is non-singular on $\mathfrak{P}$. Thus $\mathfrak{Y} = yU(\mathfrak{L}) = (0)$, and the proof is complete.

THEOREM 1. *Let $\mathfrak{L}$ be a nilpotent Lie algebra over an arbitrary field K . If $x \rightarrow U_x$ is an indecomposable representation of $\mathfrak{L}$ of finite degree, then any two irreducible constituents of U are equivalent.*

PROOF. Let $x \rightarrow R_x$ and $x \rightarrow S_x$ be two irreducible constituents of U . If $\mathfrak{B}$ is a representation space of U , then R and S must appear as representations induced upon two composition factors of $\mathfrak{B}$ arising from a single composition series of $\mathfrak{B}$. It follows that, for each $x \in L$, the minimum polynomials h_x and k_x of R_x and S_x respectively must divide the minimum polynomial of U_x , which is a power of an irreducible polynomial by Proposition 1. Therefore the minimum polynomials h_x and k_x are powers of the same irreducible polynomial in $K[X]$ and, over an algebraic closure of K , will have identical sets of distinct roots.

If we now extend the base field to K' , the algebraic closure of K , then h_x and k_x remain the minimum polynomials of the extended l.t. R'_x and S'_x . Let e and f be the degrees (over K') of the representations R' and S' , and let $x \rightarrow T'_x$ be the Kronecker sum representation $x \rightarrow R'_x \otimes 1_f + 1_e \otimes -(S'_x)^*$ of the representations R' and $-S'^*$, where S'^* is the transpose of S' . Then by an argument given by Zassenhaus [10, p. 154], zero must appear as a characteristic value of each of the l.t. T'_x . Therefore $\det T'_x = 0$, $x \in \mathfrak{L}$, and by Proposition 2, since K' is an infinite field, there exists a nonzero vector $w' \in \mathfrak{B}'$, where $\mathfrak{B}'$ is a representation space for T' , such that $w'T'(\mathfrak{L}) = (0)$.

The representation space of R' is $\mathfrak{B} \otimes K'$, while that of S' is $\mathfrak{B} \otimes K'$, where $\mathfrak{B}$ and $\mathfrak{B}$ are representation spaces of R and S respectively. The representation space of T' is therefore $(\mathfrak{B} \otimes K') \otimes (\mathfrak{B} \otimes K')$, which is isomorphic to $(\mathfrak{B} \otimes \mathfrak{B}) \otimes K'$. Moreover, since $R' = R \otimes 1$,

and $S' = S \otimes 1$, where 1 is the identity element of K' , we have

$$\begin{aligned} T' &= (R \otimes 1) \otimes 1_f + 1_e \otimes (-S^* \otimes 1) \\ &= (R \otimes 1_f + 1_e \otimes -S^*) \otimes 1. \end{aligned}$$

Thus T' is obtained by extension of the base field from the representation $T = R \otimes 1_f + 1_e \otimes -S^*$, and the representation space of T' is the extension by K' of the representation space $\mathfrak{B} \otimes \mathfrak{B}$ of T . If we write $w' = \sum w_i \lambda_i$, $\lambda_i \in K'$, where the λ_i are linearly independent over K , then we have $w' T'_x = \sum (w_i T_x) \lambda_i = 0$ for each $x \in \mathfrak{L}$, and hence each vector $w_i T_x = 0$. In particular, if $w_1 \neq 0$ then $w_1 T(\mathfrak{L}) = (0)$, and an application of Schur's Lemma yields the result that R and S are equivalent [10, p. 154].

PROPOSITION 3. *Let $\mathfrak{S}$ be an indecomposable set of l.t. in a finite-dimensional space $\mathfrak{B}$, let $\mathfrak{A}$ be the associative algebra generated by the elements of $\mathfrak{S}$ and the identity l.t., and let $\mathfrak{N}$ be the radical of $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{N}$ is simple if and only if any two of the composition factors of $\mathfrak{B}$ relative to $\mathfrak{S}$ are $\mathfrak{S}$ -isomorphic.*

PROOF. Let $\mathfrak{B} = \mathfrak{B}_0 \supset \mathfrak{B}_1 \supset \cdots \supset \mathfrak{B}_{t+1} = (0)$ be a composition series of $\mathfrak{B}$ relative to $\mathfrak{S}$; then this will be a composition series of $\mathfrak{B}$ relative to $\mathfrak{A}$. Suppose first that the composition factors $\mathfrak{B}_i - \mathfrak{B}_{i+1}$ are $\mathfrak{S}$ -isomorphic, and hence $\mathfrak{A}$ -isomorphic. Then $\mathfrak{B}_i$ is an irreducible $\mathfrak{A}$ -module such that $\mathfrak{B}_i \mathfrak{A} \neq (0)$, hence $\mathfrak{B}_i \mathfrak{N} = (0)$, and $\mathfrak{B}_i$ is an irreducible $\mathfrak{A}/\mathfrak{N}$ -module. Suppose that for some $a \in \mathfrak{A}$, $\mathfrak{B}_i a = (0)$. Then for any $x \in \mathfrak{A}$, $\mathfrak{B}_i a x = (0)$. For each i , there exists an $\mathfrak{A}$ -isomorphism E_i of $\mathfrak{B}_i$ onto $\mathfrak{B}_i - \mathfrak{B}_{i+1}$. Then $(\mathfrak{B}_i - \mathfrak{B}_{i+1}) a x = (\mathfrak{B}_i E_i) a x = (\mathfrak{B}_i a x) E_i = (0)$, $i = 0, 1, \cdots, t$, and $\mathfrak{B}_i a x \subseteq \mathfrak{B}_{i+1}$, $i = 0, 1, \cdots, t$. Thus $(a x)^{t+1} = 0$, and since x is arbitrary in $\mathfrak{A}$, $a \in \mathfrak{N}$. Therefore $\mathfrak{B}_i$ is a faithful irreducible $\mathfrak{A}/\mathfrak{N}$ -module, and $\mathfrak{A}/\mathfrak{N}$ is a simple algebra.

Conversely, if $\mathfrak{A}/\mathfrak{N}$ is simple, then since $\mathfrak{A}$ contains the identity mapping, the irreducible factors of $\mathfrak{B}$ are isomorphic to minimal right ideals in $\mathfrak{A}/\mathfrak{N}$, and hence are $\mathfrak{A}$ -isomorphic, and, a fortiori, $\mathfrak{S}$ -isomorphic.

COROLLARY. *Let $x \rightarrow U_x$ be an indecomposable representation of a nilpotent Lie algebra $\mathfrak{L}$ over an arbitrary field K . Then the associative algebra generated by the transformation U_x and the identity transformation is a primary algebra.*

The proof is a direct application of Theorem 1 and Proposition 3.

3. Extensions of irreducible representations. If $\mathfrak{L}$ is a Lie algebra over K , and if $\mathfrak{D}$ is a Lie algebra of derivations of $\mathfrak{L}$, then the semi-

direct sum $\mathfrak{L} + \mathfrak{D}$ is the Lie algebra whose underlying vector space is the direct sum of the vector spaces $\mathfrak{L}$ and $\mathfrak{D}$, and in which the bracket operation is defined by the formula

$$[x + D, x' + D'] = [x, x'] + xD' - x'D + [D, D'],$$

$$x, x' \in \mathfrak{L}; D, D' \in \mathfrak{D}.$$

Then it is well known that $\mathfrak{L} + \mathfrak{D}$ is a Lie algebra containing $\mathfrak{L}$ as an ideal.

PROPOSITION 4. *Let $\mathfrak{L}$ be a Lie algebra over an arbitrary field K , and let $\mathfrak{D}$ be a Lie algebra of derivations of $\mathfrak{L}$. If $x \rightarrow U_x$ is an irreducible representation of $\mathfrak{L}$ with representation space $\mathfrak{V}$, then there exists an irreducible representation $x \rightarrow T_x$ of $\mathfrak{L} + \mathfrak{D}$ with representation space $\mathfrak{W}$, and a one-to-one l.t. S of $\mathfrak{V}$ into $\mathfrak{W}$ such that $U_x S = S T_x$ for all $x \in \mathfrak{L}$. Moreover, if the base field has characteristic $p > 0$, so that the degree of the representation U is finite, a representation T of finite degree can be found which satisfies the above requirements.*

PROOF. First suppose that $U(\mathfrak{L}) = (0)$; then $\mathfrak{V}$ is one-dimensional. We set $\mathfrak{W} = \mathfrak{V}$, let $D \rightarrow D'$ be any linear mapping of $\mathfrak{D}$ into the set of l.t. of $\mathfrak{W}$, and let T_{x+D} be the mapping D' in $\mathfrak{W}$. If we let S be the identity mapping in $\mathfrak{W}$, then the requirements of Proposition 4 are easily verified.

Now let $\mathfrak{V}U(\mathfrak{L}) \neq (0)$; let $\mathfrak{A}$ be the universal associative algebra of $\mathfrak{L}$, and let $\mathfrak{A}'$ be the universal associative algebra of $\mathfrak{L} + \mathfrak{D}$. If $x_1, \dots, x_n$ is a basis for $\mathfrak{L}$, $D_1, \dots, D_m$ a basis for $\mathfrak{D}$, then the standard monomials $x_1^{f_1} \dots x_n^{f_n} D_1^{f'_1} \dots D_m^{f'_m}$, e_i , $f_i \geq 0$, form a basis for $\mathfrak{A}'$. The subalgebra of $\mathfrak{A}'$ generated by the elements $1, x_1, \dots, x_n$ is isomorphic to $\mathfrak{A}$, and we shall identify this subalgebra with $\mathfrak{A}$. The representation U can be extended in a unique way to a representation of $\mathfrak{A}$ which we shall again denote by U . Since $\mathfrak{V}U(\mathfrak{L}) \neq (0)$, $\mathfrak{V}$ is a cyclic $\mathfrak{A}$ -module, and is $\mathfrak{A}$ -isomorphic to $\mathfrak{A} - \mathfrak{I}$ where $\mathfrak{I}$ is a maximal right ideal in $\mathfrak{A}$.

We prove first that the right ideal $\mathfrak{I}\mathfrak{A}' = \{ \sum x_i b_i, x_i \in \mathfrak{I}, b_i \in \mathfrak{A}' \}$ is a proper ideal in $\mathfrak{A}'$. Every element in $\mathfrak{A}'$ can be expressed uniquely in the form

$$(6) \quad a_0 + \sum a_{i_1 \dots i_m} D_1^{i_1} \dots D_m^{i_m}, \quad a_0, a_{i_1 \dots i_m} \in \mathfrak{A}, (i_1, \dots, i_m) \neq 0.$$

Then $\mathfrak{I}\mathfrak{A}'$ consists of all elements (6) such that the coefficients $a_0, a_{i_1 \dots i_m} \in \mathfrak{I}$. If $\mathfrak{I}\mathfrak{A}' = \mathfrak{A}'$, then the identity element 1 can be expressed in the form (6) with coefficients in $\mathfrak{I}$. It then follows from considerations of degree that $a_0 = 1 \in \mathfrak{I}$, contrary to the fact that $\mathfrak{I}$ is a proper ideal in $\mathfrak{A}$. Then $\mathfrak{I}\mathfrak{A}'$ can be imbedded in a maximal right

ideal $\mathfrak{Y}'$ in $\mathfrak{W}'$, and since $\mathfrak{Y}' \supseteq \mathfrak{Y}\mathfrak{W}' \supseteq \mathfrak{Y}$, $\mathfrak{Y}' \cap \mathfrak{W} \supseteq \mathfrak{Y}$. But $\mathfrak{Y}' \cap \mathfrak{W}$ is a proper right ideal in $\mathfrak{W}$, and since $\mathfrak{Y}$ is maximal, $\mathfrak{Y}' \cap \mathfrak{W} = \mathfrak{Y}$.

Then $\mathfrak{W}' - \mathfrak{Y}'$ is an irreducible $\mathfrak{W}'$ -module, and if we define $T_x: u + I' \rightarrow ux + I'$ for x restricted to the linear part of $\mathfrak{W}'$, then we obtain an irreducible representation of $\mathfrak{X} + \mathfrak{D}$ acting in the space $\mathfrak{W} = \mathfrak{W}' - \mathfrak{Y}'$. Since $\mathfrak{W} - \mathfrak{Y}$ is isomorphic to $\mathfrak{B}$, there exists a one-to-one l.t. Y of $\mathfrak{B}$ onto $\mathfrak{W} - \mathfrak{Y}$ such that $YR_x = U_x Y$ for all $x \in \mathfrak{X}$, where R_x is the mapping $u + \mathfrak{Y} \rightarrow ux + \mathfrak{Y}$. On the other hand, there is a one-to-one l.t. Z of $\mathfrak{W} - \mathfrak{Y}$ into $\mathfrak{W}' - \mathfrak{Y}'$, namely $Z: x + \mathfrak{Y} \rightarrow x + \mathfrak{Y}'$, such that $R_x Z = Z T_x$ for all $x \in \mathfrak{X}$. Then $S = YZ$ is a one-to-one l.t. of $\mathfrak{B}$ into $\mathfrak{W}$ such that $U_x S = U_x YZ = YR_x Z = YZ T_x = S T_x$ for all $x \in \mathfrak{X}$.

Finally, if K has characteristic $p > 0$, the degrees of the representations U and T are both finite, by [4, Theorem 5.1]. Proposition 4 is now completely proved.

COROLLARY (ZASSENHAUS). *Let $\mathfrak{X}$ be a Lie algebra over a field K of characteristic $p > 0$, let D be a derivation of $\mathfrak{X}$, and let $x \rightarrow U_x$ be an irreducible representation of $\mathfrak{X}$ with representation space $\mathfrak{B}$. Then there exists a representation T of $\mathfrak{X}$ of finite degree, a l.t. D' acting in the representation space $\mathfrak{W}$ of T , and a one-to-one l.t. S of $\mathfrak{B}$ into $\mathfrak{W}$ such that the following conditions are satisfied: (i) $T_{xD} = [T_x, D']$ for all $x \in \mathfrak{X}$; (ii) $\mathfrak{W}$ is irreducible relative to the set $T(\mathfrak{X}) \cup \{D'\}$; and (iii) $U_x S = S T_x$ for all $x \in \mathfrak{X}$.*

REMARK. A necessary condition that the representation T of Proposition 4 have finite degree when K has characteristic zero has been obtained by Zassenhaus [11, p. 252].

PROPOSITION 5. *Let $\mathfrak{X}$ be a Lie algebra over a field K of characteristic $p > 0$, and let D be a nilpotent derivation in L . Let U be an irreducible representation of $\mathfrak{X}$, and let f be an arbitrary irreducible polynomial in $K[X]$. Then there exists a representation T of finite degree which satisfies conditions (i), (ii), and (iii) of the corollary to Proposition 4, and the further condition that the minimum polynomial of D' is a power of f .*

PROOF. First suppose that $\mathfrak{B}U(\mathfrak{X}) = (0)$. Then $\mathfrak{B}$ is one-dimensional. If f has degree m , then we can construct an m -dimensional vector space $\mathfrak{W}$ and a l.t. D' in $\mathfrak{W}$ which has f for its minimum polynomial. We define $T_x = 0$, $x \in \mathfrak{X}$; then T and $\mathfrak{W}$ satisfy conditions (i), (ii), and (iii) of the corollary to Proposition 4, where S is any one-to-one mapping of $\mathfrak{B}$ into $\mathfrak{W}$.

Now let $\mathfrak{B}U(\mathfrak{X}) \neq (0)$. We shall adopt the notation of Proposition 4, and in particular, we shall make use of the fact that $\mathfrak{B}$ is $\mathfrak{X}$ -isomorphic

to $\mathfrak{A} - \mathfrak{Z}$, where $\mathfrak{Z}$ is a certain maximal right ideal in $\mathfrak{A}$. Let $\mathfrak{A}'$ be the universal associative algebra of the semi-direct sum $\mathfrak{L} + (D)$. Since D is nilpotent, some p^k th power of it is zero, and it then follows, by a direct application of (3), that D^{p^k} is in the center of $\mathfrak{A}'$. We prove now that the right ideal $\mathfrak{B} = \mathfrak{Z}\mathfrak{A}' + g(D)\mathfrak{A}'$ is a proper right ideal in $\mathfrak{A}'$, where $g = f^{p^k}$. If it is not, then since $g(D)$ is in the center, we must have a relation

$$1 = \sum_{i=0}^r a_i D^i + \left(\sum_{j=0}^s b_j D^j \right) g(D), \quad a_i \in \mathfrak{Z}, b_j \in \mathfrak{A}.$$

We may assume that no $b_j \in \mathfrak{Z}$, for such a term could be absorbed in the left-hand summand. Then upon comparing the coefficients of the highest power of D appearing in both summands, we arrive at a contradiction.

Then $\mathfrak{B}$ can be imbedded in a maximal right ideal $\mathfrak{Z}'$, and it follows, as in the proof of Proposition 4, that $\mathfrak{Z}' \cap \mathfrak{A} = \mathfrak{Z}$, and that the representation T constructed as in Proposition 4 will satisfy the conditions of the corollary. Finally, since $\mathfrak{A}'g(D) = g(D)\mathfrak{A}' \subseteq \mathfrak{Z}'$, $g(D) \in (\mathfrak{Z}':\mathfrak{A}')$, the annihilator of the irreducible $\mathfrak{A}'$ -module $\mathfrak{A}' - \mathfrak{Z}'$. Thus the minimum polynomial of D' must divide $g = f^{p^k}$, and is consequently a power of f .

4. Irreducible representations of nilpotent Lie algebras of characteristic $p > 0$. Following Zassenhaus [9], we call a basis $(a_1, \dots, a_n)$ of a Lie algebra a *regular basis* if whenever $i < j$, $[a_i a_j] \in (a_{i+1}, \dots, a_n)$. It is easily proved by induction on the dimension that every nilpotent Lie algebra has a regular basis.

THEOREM 2. *Let $\mathfrak{L}$ be a nilpotent Lie algebra over an arbitrary field K of characteristic $p > 0$, and let $(a_1, \dots, a_n)$ be a regular basis of $\mathfrak{L}$. Let $(f_1, \dots, f_n)$ be an arbitrary ordered set of irreducible polynomials in $K[X]$. Then there exists exactly one equivalence class $\{U_x\}$ of irreducible representations of $\mathfrak{L}$ such that the minimum polynomial of U_{a_i} is a power of f_i , $i = 1, 2, \dots, n$.*

PROOF. For the existence proof, we shall use induction on n . Suppose first that $n = 1$. Then for each irreducible polynomial f of degree m , there exists a vector space $\mathfrak{B}$ of dimension m , and a cyclic l.t. A in $\mathfrak{B}$ whose minimum polynomial is f ; if $\mathfrak{L} = (a)$, then $a \rightarrow A$ is an irreducible representation of $\mathfrak{L}$. The uniqueness of A is a known result in linear algebra.

Now assume that we have an irreducible representation $x \rightarrow U_x$ of $\mathfrak{L}_1 = (a_2, \dots, a_n)$ with the required properties. Since $[a_1, \mathfrak{L}_1] \subseteq \mathfrak{L}_1$, the mapping $x \rightarrow xD = [a_1, x]$ is a nilpotent derivation in $\mathfrak{L}_1$, and ap-

plying Proposition 5, we establish the existence of an irreducible representation T such that the minimum polynomial of T_{a_1} is a power of f_1 . Let h_i be the minimum polynomial of T_{a_i} , $i=2, \dots, n$; by the corollary to Proposition 1, each h_i is a power of an irreducible polynomial. Let $\mathfrak{V}$ and $\mathfrak{W}$ be the representation spaces of U and T respectively, and let S be the one-to-one l.t. of $\mathfrak{V}$ into $\mathfrak{W}$ whose existence is guaranteed by Proposition 5. Then for each i , $2 \leq i \leq n$, the restriction of $h_i(T_{a_i})$ to the subspace $\mathfrak{V}S$ of $\mathfrak{W}$ is zero, and hence, by the induction hypothesis, the minimum polynomial of U_{a_i} , which is a power of f_i , divides h_i . Thus h_i is a power of f_i , $2 \leq i \leq n$, and the existence problem is settled.

We turn to the question of uniqueness. Let U and U' be irreducible representations of $\mathfrak{L}$, and let g_x and g'_x be the minimum polynomials of U_x and U'_x respectively, for each $x \in \mathfrak{L}$. Assume further that for $i=1, \dots, n$, the minimum polynomials g_{a_i} and g'_{a_i} are both powers of the irreducible polynomial f_i . Let $\tilde{K}$ be an algebraic closure of K ; then the extended representations $\tilde{U}$ and $\tilde{U}'$ split over $\tilde{K}$ as direct sums $\tilde{U} = T_1 + \dots + T_s$, $\tilde{U}' = T'_1 + \dots + T'_t$, where T_i and T'_j are the indecomposable constituents of $\tilde{U}$ and $\tilde{U}'$ respectively. The minimum polynomials of $\tilde{U}_x$ and $\tilde{U}'_x$ are again g_x and g'_x . We shall prove now that in $\tilde{K}$, g_x and g'_x have identical sets of distinct roots. A direct calculation shows that for each $x \in L$, $g_x = \prod_{i=1}^s g_{i,x}$, and $g'_x = \prod_{j=1}^t g'_{j,x}$, where $g_{i,x}$ and $g'_{j,x}$ are the minimum polynomials⁷ of $T_{i,x}$ and $T'_{j,x}$ respectively. It follows from the corollary to Proposition 1, applied to the case of an algebraically closed field, that for each i and x , there exist elements $\alpha_i(x)$ and $\alpha'_i(x) \in \tilde{K}$ such that $g_{i,x}(X) = (X - \alpha_i(x))^{e_{i,x}}$, and $g'_{j,x}(X) = (X - \alpha'_j(x))^{e'_{j,x}}$; the $\alpha_i(x)$ and $\alpha'_i(x)$ are the characteristic roots of the l.t. $T_{i,x}$ and $T'_{j,x}$ respectively. Our hypothesis that, for $i=1, \dots, n$, the polynomials g_{a_i} and g'_{a_i} are powers of the irreducible polynomial f_i implies that the set of distinct characteristic roots among $\alpha_1(a_i), \dots, \alpha_s(a_i)$ coincides with the set of distinct characteristic roots among $\alpha'_1(a_i), \dots, \alpha'_t(a_i)$, for $1 \leq i \leq n$. Zassenhaus has proved [10, p. 152] that the characteristic roots belonging to the elements of a regular basis, in an absolutely irreducible representation of a nilpotent Lie algebra, determine uniquely the characteristic roots of the representing l.t. of arbitrary elements of the algebra. But the elements $\alpha_i(x)$, $\alpha'_j(x)$ must coincide with the characteristic roots associated with x in the irreducible constituents of T_i and T'_j respectively. By Zassenhaus' result, and our remark con-

⁷ To prove this result, we use the fact that the $g_{i,x}$ and $g'_{j,x}$ are powers of irreducible polynomials. Then it is proved, more precisely, that g_x and g'_x are the products of the distinct $g_{i,x}$ and $g'_{j,x}$ respectively. It is this fact that is required in the proof.

cerning the distinct characteristic roots associated with a regular basis, we conclude that the set of distinct characteristic roots among $\alpha_1(x), \dots, \alpha_s(x)$ coincides with the set of distinct characteristic roots among $\alpha'_1(x), \dots, \alpha'_t(x)$ for every $x \in \mathfrak{L}$, and hence that g_x and g'_x have identical sets of distinct roots in $\bar{K}$. This implies, as we have shown in the proof of Theorem 1, that U and U' are equivalent.

COROLLARY. *There exist infinitely many inequivalent irreducible representations of a nilpotent Lie algebra over an arbitrary field of characteristic $p > 0$.*

This result follows from Theorem 2 and the fact that if K is a field, then $K[X]$ contains infinitely many distinct irreducible polynomials.

EXAMPLE. As an illustration of the theory of the u -algebra of a restricted Lie algebra of characteristic $p > 0$, N. Jacobson proved [6, p. 20] that if $\mathfrak{L}$ is the restricted Lie algebra with basis x, y, z such that $[xy] = z$, $[xz] = [yz] = 0$, $x^p = y^p = z^p = z$, then $\mathfrak{L}$ has exactly $p-1$ inequivalent restricted representations. Since $\mathfrak{L}$ is a nilpotent Lie algebra, we can establish this result as a consequence of Theorem 2. In fact, if U is an irreducible restricted representation, then since $U_z^p - U_z = 0$, the minimum polynomial f_z of U_z is a prime power factor of $X^p - X$, hence $f_z(X) = X - \alpha$, where α is an element of the prime field. Therefore $U_z = \alpha I$, where I is the identity l.t. If $\alpha = 0$ then it is immediate that $U(\mathfrak{L})$ generates a nilpotent algebra of l.t., and hence $U(\mathfrak{L}) = (0)$. For each $\alpha \neq 0$, the minimum polynomials of U_x and U_y are factors of $X^p - \alpha$, and hence are powers of $X - \alpha^{1/p}$. By the uniqueness part of Theorem 2, there are exactly $p-1$ inequivalent irreducible restricted representations.

THEOREM 3. *Let $\mathfrak{L}, a_1, \dots, a_n; f_1, \dots, f_n$ be as in Theorem 2. Then there exists a sequence k_i of positive integers such that $\lim_{i \rightarrow \infty} k_i = \infty$, and indecomposable representations $U^{(i)}$ of degree k_i , $i = 1, 2, \dots$, such that for each i , the minimum polynomial of $U_{a_j}^{(i)}$ is a power of f_j , $1 \leq j \leq n$.*

PROOF. Let U_x be the irreducible representation of $\mathfrak{L}$ belonging to $(f_1, \dots, f_n)$ according to Theorem 2, and let $\mathfrak{B}$ be the kernel of the representation corresponding to U of the universal associative algebra $\mathfrak{A}$ of $\mathfrak{L}$. We shall prove first that for each i , $\mathfrak{B}^{i+1}$ is properly contained in $\mathfrak{B}^i$. Since $\mathfrak{A}$ contains no divisors of zero, so that no finite power of $\mathfrak{B}$ is zero, it will be sufficient to prove that $\bigcap_{i=1}^{\infty} \mathfrak{B}^i = (0)$. As in the proof of Proposition 5, it follows that for a fixed integer $m = p^r$, $f_1(a_1)^m, \dots, f_n(a_n)^m$ are in the center $\mathfrak{Z}$ of $\mathfrak{A}$, and hence the $f_i(a_i)^m \in \mathfrak{B}$, $1 \leq i \leq n$. Let $\mathfrak{m}$ be the ideal in $\mathfrak{Z}$ generated

by the $f_i(a_i)^m$, $1 \leq i \leq n$; then $m\mathfrak{A} \subseteq \mathfrak{B}$. Suppose now that $\mathfrak{B}_1$ is another primitive ideal in $\mathfrak{A}$ dividing $m\mathfrak{A}$. If U^* is the irreducible representation with kernel $\mathfrak{B}_1$, then since $\mathfrak{B}_1 \supseteq m\mathfrak{A}$, the minimum polynomial of each U^*_i is a power of f_i , $1 \leq i \leq n$. By Theorem 2, the restriction of U^* to $\mathfrak{L}$ is equivalent to U , hence the kernels of the corresponding representations of $\mathfrak{A}$ coincide, and $\mathfrak{B} = \mathfrak{B}_1$. Therefore $\mathfrak{B}$ is the unique primitive ideal dividing $m\mathfrak{A}$, so that the radical of $\mathfrak{A}/m\mathfrak{A}$ is $\mathfrak{B}/m\mathfrak{A}$. Now we prove that $\mathfrak{A}/m\mathfrak{A}$ is finite-dimensional over K . Let $d_1, \dots, d_n$ be the degrees of $f_1, \dots, f_n$ respectively. Then for any standard monomial $a_1^{e_1} \dots a_n^{e_n}$ we have

$$a_1^{e_1} \dots a_n^{e_n} \equiv \sum_{0 \leq i_j \leq d_j m, 1 \leq j \leq n} \xi_{i_1 \dots i_n} a_1^{i_1} \dots a_n^{i_n} \pmod{m\mathfrak{A}},$$

where the coefficients are in K . Therefore the cosets of the $a_1^{i_1} \dots a_n^{i_n} \pmod{m\mathfrak{A}}$, $0 \leq i_j \leq d_j m$, form a set of generators of $\mathfrak{A}/m\mathfrak{A}$, and the dimension of $\mathfrak{A}/m\mathfrak{A}$ is finite. Then $\mathfrak{B}^k \subseteq m\mathfrak{A}$ for some positive integer k , and the intersection of the powers of $\mathfrak{B}$ will be zero if we can prove the corresponding result for $m\mathfrak{A}$. Let $\mathfrak{n} = \bigcap_{i=1}^{\infty} (m\mathfrak{A})^i$. Since $\mathfrak{A}$ is a finitely generated $\mathfrak{B}$ -module, where $\mathfrak{B}$ is a Noetherian ring [4], the additive theory of $\mathfrak{B}$ -submodules of $\mathfrak{A}$ due to Grundy and Snapper is applicable, and can be used to prove that $m\mathfrak{n} = \mathfrak{n}$ [5]. It then follows, exactly as in the commutative case, that since $\mathfrak{A}$ contains no divisors of zero, $\mathfrak{n} = 0$.⁸

Now we have established that $\mathfrak{B} \supset \mathfrak{B}^2 \supset \dots$, and on the other hand, each $\mathfrak{A}/\mathfrak{B}^r$ is a finite-dimensional algebra [11, p. 263]. Since $\mathfrak{A}/\mathfrak{B} \cong (\mathfrak{A}/\mathfrak{B}^r)/(\mathfrak{B}/\mathfrak{B}^r)$, $r = 1, 2, \dots$, $\mathfrak{A}/\mathfrak{B}^r$ is a primary algebra with simple residue class algebra $\mathfrak{A}/\mathfrak{B}$. If h_i is the dimension of $\mathfrak{A}/\mathfrak{B}^i$ over K , then the fact that the powers of $\mathfrak{B}$ form a properly decreasing sequence implies that $\lim_{i \rightarrow \infty} h_i = \infty$. Let $\mathfrak{I}_i$ be an indecomposable right ideal in $\mathfrak{A}/\mathfrak{B}^i$, $i = 1, 2, \dots$; if k_i is the dimension of $\mathfrak{I}_i$ over K , then since h_i/k_i is equal to the number of minimal right ideals in a direct decomposition of $\mathfrak{A}/\mathfrak{B}^i$,⁹ $\lim_{i \rightarrow \infty} k_i = \infty$. Then the representation of $\mathfrak{A}$ given by $R_{x\nu_i}$, where ν_i is the natural homomorphism of $\mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{B}^i$, and $R_{x\nu_i}$ is the mapping $u \rightarrow u(x\nu_i)$ in $\mathfrak{I}_i$, defines an indecomposable representation of $\mathfrak{A}$ of degree k_i . Finally we note that since ν_i annihilates some power of each $f_i(a_i)$, $1 \leq i \leq n$, it follows that the minimum polynomial of $R_{a_j\nu_i}$ is a power of f_j , $1 \leq j \leq n$, and $i = 1, 2, \dots$. The representations $U^{(i)}$ required by the statement of the theorem are then simply the restrictions of $R_{x\nu_i}$ to $\mathfrak{L}$, and the theorem is proved.

⁸ Cf. A. Weil, *Foundations of algebraic geometry*, New York, 1946, p. 45.

⁹ Cf. E. Artin, C. Nesbitt, and R. Thrall, *Rings with minimum condition*, Ann Arbor, 1945, Chapter 9.

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