ON UNIFORM APPROXIMATION TO CONTINUOUS FUNCTIONS BY RATIONAL FUNCTIONS WITH PREASSIGNED POLES

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In the study of uniform approximation by rational functions to functions of a complex variable, a controlling role is played by the point set of the z-plane on which the approximating takes place. In 1936, M. Lavrentieff [1] proved the following theorem:

THEOREM. *If E is a closed bounded point set of the z-plane which has no interior and does not separate the plane, then every function $f(z)$ continuous on E is uniformly approximable on E by a polynomial.*

This theorem represents the best result that can be established for uniform polynomial approximation to functions which are merely continuous; the converse may be found in J. L. Walsh's work on approximation [5]. The aim of this paper is to extend Lavrentieff's theorem by establishing it on a certain class of sets which do separate the plane. The methods used will be similar to those first introduced by M. Keldysh [2] and S. Mergelyan [3].

That the possibility of uniform approximation to continuous functions cannot be established for all closed bounded sets with no interior may be seen from the following counterexample, presented by V. A. Tonyan [4]:

Let E be the closed set with no interior obtained by removing from the unit disk a countable number of open disks k_ν which are distributed everywhere densely in the disk, and whose circumferences Γ_ν are tangent neither to each other nor to the unit circumference. Let us further choose these disks in such a way that their radii ϵ_ν satisfy $\sum_{\nu=1}^{\infty} \epsilon_\nu < \epsilon$ where $\epsilon > 0$ is a given fixed number.

Now suppose that for any function $f(z)$ continuous on E there exists a rational function $R(z)$ such that

$$\text{(a)} \qquad | R(z) - f(z) | < \epsilon \qquad \text{for } z \in E.$$

Then, denoting by k_{ν_i} $(i=1, 2, \cdots, p)$ the disks which contain the poles of $R(z)$ lying in $|z| < 1$, we have

$$\int_{\Gamma_{\nu_1}+\cdots+\Gamma_{\nu_p}} R(z)dz = \int_{|z|=1} R(z)dz.$$

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Now

$$\int_{|z|=1} f(z)dz = \int_{|z|=1} f(z) - R(z)dz + \int_{\Gamma_{\nu_1}+\cdots+\Gamma_{\nu}} R(z) - f(z)dz$$
$$+ \int_{\Gamma_{\nu_1}+\cdots+\Gamma_{\nu_p}} f(z)dz$$

and so, using (a) and the choice of the radii of the k_ν, we have

$$\text{(b)} \qquad \left| \int_{|z|=1} f(z)dz \right| \leqq 2\pi\epsilon + 2\pi\epsilon^2 + 2\pi\epsilon \max_{z \epsilon \Gamma_{\nu_i}} |f(z)|.$$

For any $f(z)$ bounded on $|z| < 1$, the right-hand side of (b) may be made arbitrarily small by appropriate choice of $\epsilon > 0$. It suffices then to pick an $f(z)$ such that $\left|\int_{|z|=1} f(z)dz\right| \neq 0$ to obtain a contradiction.

We proceed now to characterize a particular class of sets on which the possibility of uniform approximation can be established.

Definition. Given a closed bounded set E which divides the plane. Any open set C such that $E \cap$(the complement of C) no longer divides the plane will be called a *cutting set of* E.

Lemma. *Given a closed bounded set E which divides the plane, there exist cutting sets of E of arbitrarily small measure.*

Since E is bounded, it cannot divide the plane into more than a denumerable number of regions. Let us enumerate these regions. Connect region 1 with region 2 by an open set of measure $< m/2$; connect region 2 with region 3 by an open set of measure $< m/2^2$; etc. Let C be the union of these open sets. q.e.d.

Theorem. *Given E a closed bounded point set with no interior and the points $z_1, z_2, \cdots$, at least one in each of the regions into which E divides the plane. If there exist cutting sets of E whose closures have arbitrarily small measure, then any function $f(z)$ continuous on E may be uniformly approximated on E by a rational function with poles in the points z_k.*

It may be pointed out that if the bounded set E has measure zero it automatically satisfies the conditions of the theorem.

Proof. Since, by Weierstrass' theorem, any continuous function on a closed set may be uniformly approximated by polynomials $\Pi(x, y)$ in x and y, it is sufficient to prove the theorem for such polynomials. Let $\Pi(x, y) = u(x, y) + iv(x, y)$, u, v real functions.

Let us choose a sequence of open sets $D_1, D_2, \cdots$ so that:

1. $E \in D_{n+1} \in D_n$.

2. Every D_n consists of a finite number of disjoint regions, each with an analytic Jordan boundary.

3. The complements of $\overline{D}_n$ converge to the complement of E.

Let $L_n = \overline{D}_n - D_n$. Now from Green's formula we have

$$\Pi(x, y) = \Pi(z) = \frac{1}{2\pi i} \int_{L_n} \frac{\Pi(t)}{t - z}\, dt + \frac{1}{2\pi} \iint_{\overline{D}_n} \frac{u_\xi - v_\eta + i(u_\eta + v_\xi)}{\zeta - z}\, d\xi d\eta$$

where $u_\xi = \partial u(\xi, \eta)/\partial \xi$, $\zeta = \xi + i\eta$, $z = x + iy$, $z \in E$, and for $F(\zeta) = u_\xi - v_\eta + i(u_\eta + v_\xi) = \Pi_\xi + i\Pi_\eta$ we have $|F(\zeta)| < M$, $\zeta \in E$. Since

$$L(z) = \frac{1}{2\pi i} \int_{L_n} \frac{\Pi(t)}{t - z}\, dt$$

is analytic for $z \in E$ and since

$$\iint_{\overline{D}_n} \frac{F(\zeta)}{\zeta - z}\, d\xi d\eta, \qquad z \in E,$$

approaches uniformly

$$\iint_E \frac{F(\zeta)}{\zeta - z}\, d\xi d\eta, \qquad z \in E,$$

as $n \to \infty$, it is sufficient to show, by virtue of the boundedness of $F(\zeta)$, that given $\epsilon' > 0$

$$\max_{z \in E} \left| \iint_{\zeta \in E} \frac{1}{\zeta - z}\, d\xi d\eta - q(z) \right| < \epsilon'$$

for some $q(z)$ analytic for $z \in E$.

Given $m > 0$, by hypothesis there exists a cutting set C of E such that the measure of $\overline{C}$ is $< m$. We can therefore find two open sets A, B such that $\overline{C} \subset B$, $\overline{B} \subset A$, and the measure of $A < 2m$. Let $2d$ be the least distance between the boundary of C and that of B; $2d > 0$. Let C_d, C_{2d}, be C plus the set of all points exterior to C whose distance from the boundary of C is $\leqq d$, $2d$ respectively. Let

$$f(z) = \iint_{\zeta \in E} \frac{1}{\zeta - z}\, d\xi d\eta, \qquad z \in E,$$

$$g(z) = \iint_{\zeta \in E \cap [\text{complement of } A]} \frac{1}{\zeta - z}\, d\xi d\eta, \qquad z \in E.$$

Now

$$(1^\circ) \qquad |f(z) - g(z)| < \iint_{\zeta\in E\cap A} \frac{1}{|\zeta - z|}\, d\xi d\eta < A_1 m^{1/2}, \qquad z \in E,$$

so that it will be sufficient to approximate uniformly to $g(z)$, $z\in E$.

For $z\in E\cap$[compl. of C], $g(z)$ is continuous and $E\cap$[compl. of C] is a closed set with no interior not dividing the plane. Therefore by Lavrentieff's theorem, we can find a polynomial $p(z)$ such that

$$(1) \qquad |g(z) - p(z)| < \epsilon, \qquad z \in E \cap [\text{compl. of } C].$$

For $z\in A$, $g(z)$ is analytic, and so on the closed set $\overline{B}\cap E$ we can find an analytic function $a(z)$ such that

$$(2) \qquad |g(z) - a(z)| < \epsilon, \qquad z \in E \cap \overline{B}.$$

Now let

$$\phi(z, s) = \begin{cases} p(z) & \text{for } s \in [\text{compl. of } C_d], \\ a(z) & \text{for } s \in C_d, \end{cases} \qquad z \in E.$$

In any circle $D(s)$: $|z-s| <d$, $z\in E$, we have, by (1) and (2),

$$(3) \qquad |g(z) - \phi(z, s)| < \epsilon, \qquad z \in E.$$

Let us now introduce

$$K(r) = \begin{cases} \dfrac{3}{\pi d^2}\left(1 - \dfrac{r}{d}\right), & 0 < r \leqq d, \\ 0, & r > d, \end{cases}$$

and let

$$\phi(z) = \iint_{|s|<2R_0} \phi(z, s) K(|s - z|)\, du dv, \qquad z \in E,$$

where $s=u+iv$, $z=x+iy$, R_0 is the maximum diameter of E. We now observe some of the properties of the smoothing function K:

$$\text{(A)} \quad \iint_{|s|<2R_0} K(|s - z|)\, du dv = 1,$$

$$\text{(B)} \quad \iint_{|s|<2R_0} K_x(|s - z|)\, du dv = \iint_{|s|<2R_0} K_y(|s - z|)\, du dv = 0,$$

$$\text{(C)} \quad \left.\begin{aligned} &\iint_{|s|<2R_0} |K_x(|s - z|)|\, du dv \\ &\iint_{|s|<2R_0} |K_y(|s - z|)|\, du dv \end{aligned}\right\} < \frac{6}{d}\,.$$

Consequently, by (A) and (3),

$$\begin{aligned} &|\phi(z) - g(z)| \\ (4) \qquad &\leqq \iint_{|s|<2R_0} |\phi(z, s) - g(z)| K(|s - z|)dudv < \epsilon, \quad z \in E. \end{aligned}$$

Let us now examine more of the properties of $\phi(z)$. For z outside $C_{2d}-C$, $z\in E$, $\phi(z)=p(z)$ or $a(z)$, depending on whether z is outside or inside of C. Therefore, at those points, $\psi(z)\equiv\phi_x(z)+i\phi_y(z)=0$ by the Cauchy-Riemann equations.

For z in $(C_{2d}-C)\cap E$

$$\begin{aligned} \psi(z) = &\iint_{|s|<2R_0} [\phi_x(z, s) + i\phi_y(z, s)]K(|s - z|)dudv \\ &+ \iint_{|s|<2R_0} \phi(z, s)[K_x(|s - z|) + iK_y(|s - z|)]dudv. \end{aligned}$$

$\iint_{|s|<2R_0}[\phi_x(z, s)+i\phi_y(z, s)]K(|s-z|)dudv=0$ since, for every point $s\in C_d$, $\phi(z, s)$ is analytic for $z\in E$ $[=a(z)]$, thus $\phi_x+i\phi_y=0$, and for every point $s\in$[complement of C_d], $\phi(z, s)$ is analytic for $z\in E[=p(z)]$, thus $\phi_x+i\phi_y=0$. Finally, therefore,

$$\begin{aligned} \psi(z) &= \iint_{|s|<2R} \phi(z, s)[K_x(|s - z|) + iK_y(|s - z|)]dudv \\ &\qquad\qquad\qquad\qquad (z \in (C_{2d} - C) \cap E) \\ &= 0 \qquad\qquad\qquad (z \text{ outside } C_{2d} - C,\ z \in E) \end{aligned}$$

and, using (B), (C), and (3), we obtain

$$|\psi(z)| = \left|\iint_{|s|<2R_0} [\phi(z, s) - g(z)][K_x + iK_y]dudv\right| \leqq \frac{12\epsilon}{d}, \ z \in E.$$

We know that the inequalities (1), (2), and consequently (3) and (4), as well as the analyticity of $a(z)$, hold not only on E but on a somewhat larger set which includes E in its interior. Therefore this is true as well of the properties which we have derived for ψ. Let us call by $R(\epsilon)$ the larger open set which includes E in its interior and such that on $\overline{R(\epsilon)}$ all the relations we have established for $z\in E$ still persist. Then if T is an open set with a sufficiently smooth boundary which lies interior to $R(\epsilon)$ and which contains E (we may take T to be one of the sets D_n previously constructed with n sufficiently large) we have, by Green's formula again,

$$\phi(z) = \frac{1}{2\pi i}\int_{\text{Boundary of } T} \frac{\phi(t)}{t - z}\,dt + \frac{1}{2\pi}\iint_{\zeta\in T} \frac{\psi(\zeta)}{\zeta - z}\,d\xi d\eta, \qquad z\in E;$$

by virtue of what we know about $\psi(\zeta)$, we may write

$$\phi(z) = \frac{1}{2\pi i}\int_{\text{Boundary of } T} \frac{\phi(t)}{t - z}\,dt + \frac{1}{2\pi}\iint_{\zeta\in [C_{2d}-C]\cap T} \frac{\psi(\zeta)}{\zeta - z}\,d\xi d\eta, \qquad z\in E.$$

The first integral is analytic for $z\in E$; thus there exists a function $q(z)$ analytic for $z\in E$ such that

$$\left|\phi(z) - q(z)\right| \leqq \frac{1}{2\pi}\iint_{\zeta\in [C_{2d}-C]\cap T} \frac{\left|\psi(\zeta)\right|}{\left|\zeta - z\right|}\,d\xi d\eta \leqq \frac{12\epsilon}{2\pi d} A_2 m^{1/2}, \quad z\in E.$$

Using (4) and (1°) we now have

$$\left|f(z) - q(z)\right| < A_1 m^{1/2} + \epsilon + \frac{\epsilon}{d} A_2 m^{1/2}, \qquad z\in E,$$

A_1, A_2 absolute constants. Given $\epsilon' > 0$, choose m small enough to make $A_1 m^{1/2} < \epsilon'/2$; this determines d. Then pick ϵ sufficiently small to make the rest $< \epsilon'/2$. q.e.d.

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