SOME CRITERIA OF UNIVALENCE[1]

ZEEV NEHARI

In an earlier paper [3] it was shown that the univalence of an analytic function $w=f(z)$ in the unit disk can be assured by conditions of the type $|\{w, z\}| \leqq m(|z|)$, where $m(|z|)$ is a suitable positive function and

$$\{w, z\} = \left(\frac{w''}{w'}\right)' - \frac{1}{2}\left(\frac{w''}{w'}\right)^2$$

is the Schwarzian derivative of $w=f(z)$. The two cases treated in [3] were $m(|z|)\equiv\pi^2/2$ and $m(|z|)=2(1-|z|^2)^{-2}$. The constants appearing in both criteria are the largest possible. In the first case this is shown by the existence of the nonunivalent function $w=\tan \pi(1+\epsilon)z/2$ $(\epsilon>0)$ for which $\{w, z\}=\pi^2(1+\epsilon)^2/2$, and in the second case by an example constructed by E. Hille [2]. Other criteria of this type have meanwhile been announced (without proof) by V. Pokornyi [4], the only sharp one among them being the one corresponding to $m(|z|) =4(1-|z|^2)^{-1}$, with the extremal $f(z)=\int_0^z(1-z^2)^{-2}dz$.

The main objective of the present note is to establish the following more general criterion of univalence.

THEOREM I. *The function $f(z)$ will be univalent in $|z|<1$ if*

$$(1) \qquad |\{f(z), z\}| \leqq 2p(|z|),$$

where $p(x)$ is a function with the following properties: (a) *$p(x)$ is positive and continuous for $-1<x<1$*; (b) $p(-x)=p(x)$; (c) $(1-x^2)^2p(x)$ *is nonincreasing if x varies from 0 to 1*; (d) *the differential equation*

$$(2) \qquad y''(x) + p(x)y(x) = 0$$

has a solution which does not vanish for $-1<x<1$. The constant 2 in (1) *cannot be replaced by a larger number.*

The proof of Theorem I, like that of the other criteria mentioned above, rests on the fact that a function $f(z)$ is univalent in a region D if, and only if, no solution of the differential equation

$$(3) \qquad u''(z) + q(z)u(z) = 0, \qquad 2q(z) = \{f(z), z\}$$

vanishes in D more than once [3]. If $f(z)$ is not univalent in $|z|<1$,

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there will therefore be two points, say α and β ($|\alpha| < 1$, $|\beta| < 1$, $\alpha \neq \beta$), at which one of the solutions of (3) will vanish. There exists a unique circle which passes through α and β and is orthogonal to $|z| = 1$. This circle is divided by $|z| = 1$ into two arcs, one of which contains the points α, β and will be denoted by C. Since the statement of Theorem I is invariant with respect to a rotation of the z-plane about the origin we may assume, without losing generality, that C is in the upper half-plane and symmetric with respect to the imaginary axis.

A suitable linear substitution of the form[2]

$$z = \frac{w + \zeta}{1 + \zeta^* w} \qquad (|\zeta| < 1) \tag{4}$$

will carry C into the linear segment $-1 < w < 1$, and it will, of course, map $|z| < 1$ onto $|w| < 1$. It is easy to see that, because of the particular location of C, one of these substitutions must be of the form

$$z = \frac{w + i\rho}{1 - i\rho w}, \qquad 0 \leqq \rho < 1. \tag{5}$$

The points α, β are carried, respectively, into two points a, b on the real axis. We may assume, without loss of generality, that a is at the left of b, so that $-1 < a < b < 1$.

The substitution (4) will transform the equation (3) into

$$v''(w) + q_1(w)v(w) = 0, \qquad u(z) = \phi(w)v(w), \tag{6}$$

where $\phi(w)$ is regular and different from zero in $|w| < 1$, and

$$2q_1(w) = \{g(w), w\}, \qquad g(w) = f\left(\frac{w + \zeta}{1 + \zeta^* w}\right). \tag{7}$$

It is easily confirmed that

$$\{g(w), w\} = \left(\frac{dz}{dw}\right)^2 \{f(z), z\}$$

and that

$$\left|\frac{dz}{dw}\right| = \frac{1 - |z|^2}{1 - |w|^2}.$$

It follows therefore that

$$(1 - |w|^2)^2 |\{g(w), w\}| = (1 - |z|^2)^2 |\{f(z), z\}|,$$

[2] Asterisks denote complex conjugates.

and thus, by (1), that

$$(1 - |w|^2)^2 |\{g(w), w\}| \leqq 2(1 - |z|^2)^2 p(|z|).$$

By hypothesis (c) of Theorem I, $(1-x^2)^2 p(x)$ is nonincreasing if x grows from 0 to 1. Now it is evident from (5) that $|z| > |w|$ if $-1 < w < 1$. Hence,

$$(1 - |z|^2)^2 p(|z|) \leqq (1 - w^2)^2 p(w), \qquad -1 < w < 1,$$

and therefore

$$|\{g(w), w\}| \leqq 2p(w), \qquad -1 < w < 1. \tag{8}$$

By our assumptions, there exists a solution $v(w)$ of (6) which vanishes at two points a, b for which $-1 < a < b < 1$. Multiplying (6) by $v^*(w)$ and integrating from a to b along the real axis, we obtain, after an integration by parts,

$$\int_a^b |v'(w)|^2 dw = \int_a^b q_1(w) |v(w)|^2 dw.$$

Hence, by (7) and (8),

$$\int_a^b |v'(w)|^2 dw \leqq \int_a^b p(w) |v(w)|^2 dw.$$

If we write $v(w) = \sigma(w) + i\tau(w)$, both $\sigma(w)$ and $\tau(w)$ vanish for $w = a, b$, and we have $|v'(w)|^2 = \sigma'^2(w) + \tau'^2(w)$. Thus,

$$\int_a^b [\sigma'^2(w) + \tau'^2(w)] dw \leqq \int_a^b p(w) [\sigma^2(w) + \tau^2(w)] dw. \tag{9}$$

Let now λ be the lowest eigenvalue of the differential system

$$y''(w) + \lambda p(w) y(w) = 0, \qquad y(a) = y(b) = 0.$$

By Rayleigh's inequality, we have

$$\lambda \int_a^b p(w) \sigma^2(w) dw \leqq \int_a^b \sigma'^2(w) dw,$$

and a similar inequality for $\tau(w)$. Combining this with (9), we obtain

$$\int_a^b [\sigma'^2(w) + \tau'^2(w)] dw \leqq \frac{1}{\lambda} \int_a^b [\sigma'^2(w) + \tau'^2(w)] dw.$$

It follows that $\lambda \leqq 1$ and therefore, in view of $p(w) > 0$ and the Sturm comparison theorem, that a solution of (2) which vanishes at

$w=a$ must have yet another zero in the interval $a<w\leqq b$. By the Sturm separation theorem, all solutions of (2) must therefore vanish in the interval $a\leqq w\leqq b$, and thus in $-1<w<1$. But this disagrees with hypothesis (d) of Theorem I, and the assumption that there exists a solution of (3) which vanishes in $|z|<1$ more than once will therefore lead to a contradiction. It follows that the function $f(z)$ must be univalent if it satisfies the hypotheses of Theorem I.

If $p(x)\equiv\pi^2/4$, or $p(x)=(1-x^2)^{-2}$, equation (2) has the solutions $y=\cos \pi x/2$ and $y=(1-x^2)^{1/2}$, respectively, and Theorem I applies, yielding the results derived in [**3**]. Since these results are sharp, the same is thus true of Theorem I. Theorem I is, however, not only sharp in the sense that the constant 2 in (1) cannot in general be replaced by a larger one. The following, more precise, statement holds.

Let $p(z)$ be regular in $|z|<1$, $|p(z)|\leqq p(|z|)$, and let $p(x)$ $(z=x+iy)$ *satisfy hypotheses* (a), (b), (c), (d) *of Theorem* I. *If* (2) *has a solution which vanishes for $x=\pm 1$, and if $\epsilon>0$, then there exists a function $f(z)$ which is not univalent in $|z|<1$ and for which*

$$|\{f(z), z\}| = (2+\epsilon)p(|z|) \tag{10}$$

for suitable values of z.

Indeed, if we set $2q(z)=(2+\epsilon)p(z)$, the equation $u''(z)+q(z)u(z)=0$ will—by the Sturm comparison theorem—have a solution with two zeros in the interval $-1<z<1$. It follows that the function $f(z)$, for which $\{f(z), z\}=2q(z)$, cannot be univalent in $|z|<1$. Since, moreover, $f(z)$ satisfies (10) for real values of z, our statement is proved.

Every function $p(x)$ which satisfies hypotheses (a), (b), (c), and for which we can find the lowest eigenvalue λ of the differential system

$$y''(x) + \lambda p(x)y(x) = 0, \qquad y(\pm 1) = 0, \tag{11}$$

will therefore yield a sharp criterion of univalence, provided $p(z)$ is regular in $|z|<1$ and $|p(z)|\leqq p(|z|)$. For instance, if $p(x)=(1-x^2)^{-1}$, (11) has the solution $y(x)=1-x^2$, with $\lambda=2$. $f(z)$ will thus be univalent in $|z|<1$ if $|\{f(z), z\}|\leqq 4(1-|z|^2)^{-1}$, in accordance with the result of Pokornyi [**4**] mentioned further above.

If (11) cannot be solved explicitly, less accurate criteria can be obtained by estimating the eigenvalue λ from below. As an illustration, we replace (11) by the equivalent integral equation

$$y(\xi) = \lambda \int_{-1}^{1} p(x)g(x, \xi)y(x)dx, \tag{12}$$

where $2g(x, \xi) = (1+x)(1-\xi)$ for $-1 \leqq x \leqq \xi$ and $2g(x, \xi) = (1+\xi)(1-x)$ for $\xi \leqq x \leqq 1$. Obviously, $2g(x, \xi) \leqq 1 - x^2$ for $-1 \leqq x \leqq 1$. If ξ is taken to be such that $|y(x)| \leqq |y(\xi)|$ in $-1 \leqq x \leqq 1$, it follows from (12) that

$$1 \leqq \lambda \int_{-1}^{1} p(x) g(x, \xi) dx,$$

and therefore

$$2 \leqq \lambda \int_{-1}^{1} p(x)(1 - x^2) dx. \tag{13}$$

Combining this with Theorem I, we arrive at the following result.

If

$$| \{f(z), z\} | \leqq \frac{2p(|z|)}{\int_0^1 (1 - x^2) p(x) dx}, \qquad |z| < 1, \tag{14}$$

and $p(x)$ *satisfies hypotheses* (a), (b), (c) *of Theorem* I, *then* $f(z)$ *is univalent in* $|z| < 1$.

While it is known that the constant 2 in (13) is the largest possible [1], the decision whether or not (14) is the best criterion of its kind will depend on the existence—or non-existence—of functions $p(x)$ for which the right-hand side of (13) is arbitrarily close to 2 and which, at the same time, satisfy $|p(z)| \leqq p(|z|)(|z| < 1)$ and hypotheses (a), (b), (c).

References

1. P. Hartman and A. Wintner, *On an oscillation criterion of Liapounoff,* Amer. J. Math. vol. 73 (1951) pp. 885–890.

2. E. Hille, *Remarks on a paper by Zeev Nehari,* Bull. Amer. Math. Soc. vol. 55 (1949) pp. 552–553.

3. Z. Nehari, *The Schwarzian derivative and schlicht functions,* Bull. Amer. Math. Soc. vol. 55 (1949) pp. 545–551.

4. V. V. Pokornyi, *On some sufficient conditions for univalence,* Doklady Akademii Nauk SSSR (N.S.) vol. 79 (1951) pp. 743–746.

Washington University