THE REPRESENTATION OF LINEAR OPERATORS FROM L^p TO L

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A satisfactory representation theory in terms of integrals for linear operators from an L^p space to an arbitrary Banach space X has not as yet been developed for $p > 1$. For the case $p = 1$, Dunford and Pettis [3] have obtained results provided X satisfies certain rather general restrictions. In particular, the representations for the cases in which X is an L space or an L^p space are well known. The case in which the domain is L and X is an arbitrary space has been treated by Rickart [6] who obtained a representation in terms of the Phillips-Birkhoff integral [5; 2]. If the range space X is a function space in which the values of the functions at a fixed point in their common domain of definition yield a linear functional over the space, the problem can frequently be treated if the form of the linear functional over the domain space is known by considering the operator as a family of linear functionals over the space. In the present paper we use essentially this technique to obtain a representation for operators from L^p to L in terms of the Lebesgue integral.

We denote by (R, ϕ) , (S, γ) measure spaces where R, S are sets, ϕ, γ completely additive measures defined over σ -rings $\mathcal{F}(R)$, $\mathcal{F}(S)$ of subsets in R, S respectively, and we assume that R, S are unions of countable families of sets of finite measure. The set of all real-valued functions $\{x(t)\}$ defined over R which are measurable and such that $|x(t)|^p$ is integrable over R with respect to ϕ is denoted by $L^p(R, \phi)$. It is well known that if $1 \leq p < \infty$, $L^p(R, \phi)$ forms a Banach space where the algebraic operations are defined in the obvious manner and where the norm is given by $\|x\| = \{\int_R |x(t)|^p d\phi\}^{1/p}$. If $p = \infty$, the space $L^\infty(R, \phi)$ is the space of essentially bounded functions with norm $\|x\| = \text{ess sup}_t |x(t)|$. By a theorem of F. Riesz [7], the linear functional over $L^p(R, \phi)$ is representable by the expression $f(x) = \int_R h(t)x(t)d\phi$ if $1 \leq p < \infty$ where $h \in L^{p'}(R, \phi)$ for $p > 1$, p' defined by $1/p + 1/p' = 1$ and $h \in L^\infty(R, \phi)$ if $p = 1$.

We shall make extensive use of the theorem of Radon and Nikodym [8] which states that $L(R, \phi)$ is isometrically isomorphic to the space of completely additive and absolutely continuous set functions $A(R, \phi)$ defined over $\mathcal{F}(R)$ where $F \in A(R, \phi)$ implies that $\|F\|$

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$= \text{var}_{e \in \mathcal{Y}(R)} F(e)$ where we define $\text{var}_{e \in \mathcal{Y}(R)} F(e) = \sup_{e \in \mathcal{Y}(R)} F(e) - \inf_{e \in \mathcal{Y}(R)} F(e)$. The correspondence between $L(R, \phi)$ and $A(R, \phi)$ is set up by $F(e) = \int_e x(t) d\phi$ where $x(t)$ is the essentially unique point function which corresponds to $F(e)$ by the Radon-Nikodym theorem. We shall also prove and use extensively a generalization of a theorem of Riesz [7] on the equivalence for $1 \leq p < \infty$ of the spaces L^p and V^p where V^p is the space of set functions of bounded p -variation.

The space $V^p(R, \phi)$ is defined to be the space of all completely additive set functions on $\mathcal{Y}(R)$ having the property that $F \in V^p(R, \phi)$ implies the existence of a constant K depending on F such that the quantity $\sum_{i=1}^n (|F(e_i)|^p / \phi(e_i)^{p-1}) < K$ for all finite families $\{e_i\}$ of disjoint measurable subsets in $\mathcal{Y}(R)$ of finite nonzero measure. $V^p(R, \phi)$ is a Banach space if the norm of an element F is defined by

$$\|F\| = p\text{-var } F(e) = \sup_{\pi} \left\{ \sum_{i=1}^n (|F(e_i)|^p / \phi(e_i)^{p-1}) \right\}^{1/p}$$

where the upper bound is taken over all finite families of subsets of the type described above. F. Riesz showed [7] that in the case $1 \leq p < \infty$ and R the unit interval, V^p and L^p are isomorphic and isometric under the correspondence $F(e) = \int_e x(t) dt$ and that $p\text{-var } F(e) = (\int_R |x(t)|^p dt)^{1/p}$. The following proof of the same equivalence in the case of a general measure space is similar to that of Riesz for the real interval.

THEOREM. *The Banach space $V^p(R, \phi)$ is isomorphic and isometric to the space $L^p(R, \phi)$ for $1 \leq p < \infty$ where R and ϕ satisfy the conditions quoted in the introduction.*

PROOF. Let $x \in L^p(R, \phi)$. Define $f(e) = \int_e x(t) d\phi$. It must be shown that $f \in V^p(R, \phi)$. Let π be a collection of a finite number of disjoint sets $\{e_i\}$ each of finite nonzero measure, $i = 1, 2, \dots, n$. Let $E_\pi = \bigcup_{i=1}^n e_i$. By the Hölder inequality

$$\begin{aligned} \left| \int_{e_i} x(t) d\phi \right| &\leq \left(\int_{e_i} d\phi \right)^{1/p'} \left\{ \int_{e_i} |x(t)|^p d\phi \right\}^{1/p} \\ &= \{\phi(e_i)\}^{(p-1)/p} \left\{ \int_{e_i} |x(t)|^p d\phi \right\}^{1/p}, \end{aligned}$$

and

$$|f(e_i)|^p = \left| \int_{e_i} x(t) d\phi \right|^p \leq \{\phi(e_i)\}^{p-1} \int_{e_i} |x(t)|^p d\phi.$$

Hence

$$\begin{aligned} \|f\| &= \sup_{\pi} \left\{ \sum_{i=1}^n \frac{f(e_i)^p}{\phi(e_i)^{p-1}} \right\}^{1/p} \leq \left\{ \sum_{i=1}^n \int_{e_i} x(t)^p d\phi \right\}^{1/p} \\ &\leq \left\{ \int_R |x(t)|^p d\phi \right\}^{1/p} = \|x\|_p. \end{aligned}$$

This shows that $f \in V^p(R, \phi)$ and that $\|f\|_{V^p} \leq \|x\|_p$.

Conversely, let $f \in V^p(R, \phi)$. Then if e is any set of finite nonzero measure, $|f(e)|^p / \phi(e)^{p-1} \leq \|f\|^p$ and hence $|f(e)| \leq \{\|f\| [\phi(e)]^{p-1}\}^{1/p}$ which shows that f is absolutely continuous and hence can be represented as an indefinite integral by the Radon-Nikodym theorem. Let $x(t) \in L(R, \phi)$ be a function such that $f(e) = \int_e x(t) d\phi$. It must be shown that $x \in L^p(R, \phi)$ and that $\|x\|_p \leq \|f\|_{V^p}$.

Let E be a subset of R of finite measure and let $e_{jk}, j=1, 2, 3, \dots, k=0, \pm 1, \pm 2, \pm 3, \dots$ be the set of points of E for which $(k-1)/j \leq x(t) < k/j$. Each e_{jk} is measurable and of finite measure. Also if $\phi(e_{jk}) > 0$

$$\frac{k-1}{j} \leq \frac{\int_{e_{jk}} x(t) d\phi}{\phi(e_{jk})} = \frac{f(e_{jk})}{\phi(e_{jk})} < \frac{k}{j}.$$

Let

$$x_j(t) = \begin{cases} f(e_{jk})/\phi(e_{jk}) & \text{for } t \in e_{jk} \text{ if } \phi(e_{jk}) > 0, \\ 0 & \text{for } t \in e_{jk} \text{ if } \phi(e_{jk}) = 0. \end{cases}$$

Thus $\lim_{j \rightarrow \infty} x_j(t) = x(t)$ for almost all t in E . Also

$$\int_E |x_1(t)|^p d\phi = \sum_{k=-\infty}^{\infty} \frac{|f(e_{jk})|^p}{[\phi(e_{jk})]^p} \cdot \phi(e_{jk}) \leq \|f\|_{V^p}^p$$

for any j . The inequality holds for the infinite summation since the finite partial sums are uniformly bounded by the same constant. Hence $x_j \in L^p(E, \phi)$ for any E of finite measure. Since $|x(t)|^p$ is a limit almost everywhere of a sequence of summable non-negative functions, by Fatou's lemma $|x(t)|^p$ is integrable and $x \in L^p(E, \phi)$. Also

$$\left\{ \int_E |x(t)|^p d\phi \right\}^{1/p} \leq \sup_{\pi} \left\{ \sum_{i=1}^n \frac{|f(e_i)|^p}{\phi(e_i)^{p-1}} \right\}^{1/p}$$

for any partition of E into disjoint subsets of finite nonzero measure. Since the supremum on the right is uniformly bounded by $\|f\|_{V^p}$ for any such partition of R and since R is a union of countably many

measurable sets of finite measure, $\|x\|_p \leq \|f\|_{V^p}$ and this combined with the reverse inequality obtained earlier shows that $L^p(R, \phi)$ and $V^p(R, \phi)$ are isomorphic and isometric.

LEMMA. Let $\pi = \{e_i\}$, $i=1, 2, \dots, n$, be a finite family of disjoint sets in $\mathcal{F}(R)$ such that $0 < \phi(e_i) < \infty$ for each i . Let $\psi_{e_i}(t)$ be the characteristic function of the set e_i . Then the set of functions of the form

$$\sum_{i=1}^n \frac{a_i \psi_{e_i}(t)}{[\phi(e_i)]^{1/p}}$$

where the sum $\sum_{i=1}^n |a_i|^p$ does not exceed unity and where the families $\{e_i\}$ and $\{a_i\}$ range over all such families, is dense in the unit sphere of $L^p(R, \phi)$.

PROOF. Let $z(t) = \sum_{i=1}^n \{a_i \psi_{e_i}(t) / [\phi(e_i)]^{1/p}\}$ satisfy the hypotheses. Then

$$\begin{aligned} \left\{ \int_R |z(t)|^p d\phi \right\}^{1/p} &= \left\{ \sum_{i=1}^n |a_i|^p \frac{\int_{e_i} \psi_{e_i}(t) d\phi}{\phi(e_i)} \right\}^{1/p} \\ &= \left\{ \sum_{i=1}^n |a_i|^p \right\}^{1/p} \leq 1 \end{aligned}$$

and

$$\|z\| \leq 1 \quad \text{in } L^p(R, \phi).$$

A simple function is one which takes on a finite set of values, each on a set of finite measure. It is well known that the simple functions of unit norm or less are dense in the unit sphere of $L^p(R, \phi)$. If $y(t) = \sum_{i=1}^n b_i \psi_{e_i}(t)$ is of this form

$$\left\{ \int_R \left| \sum_{i=1}^n b_i \psi_{e_i}(t) \right|^p d\phi \right\}^{1/p} = \left\{ \sum_{i=1}^n |b_i|^p \phi(e_i) \right\}^{1/p}.$$

If the sequence $\{a_i\}$ is chosen to be $a_i = b_i [\phi(e_i)]^{1/p}$ then

$$\sum_{i=1}^n b_i \psi_{e_i}(t) = \sum_{i=1}^n a_i \frac{\psi_{e_i}(t)}{[\phi(e_i)]^{1/p}}$$

where $\sum_{i=1}^n |a_i|^p \leq 1$. Thus any simple function of norm one can be written in the above form and the lemma is proved.

THEOREM. If $1 < p < \infty$, a necessary and sufficient condition that T be a bounded linear operator from $L^p(R, \phi)$ to $A(S, \gamma)$ is that

$$Tx = \int_R K(e, t)x(t)d\phi$$

where the kernel function $K(e, t)$ is a real-valued function defined for $e \in \mathcal{J}(S)$, $t \in R$ satisfying the following conditions:

(a) $K(e, t) \in L^{p'}(R, \phi)$ for each $e \in \mathcal{J}(S)$, $(1/p) + (1/p') = 1$.

(b) $\int_R K(e, t)d\phi \in A(S, \gamma)$ for any $E \in \mathcal{J}(R)$, $0 < \phi(E) < \infty$.

(c) $\sup_{e \in \mathcal{J}(S)} \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'} = M < \infty$.

The norm of T satisfies the inequality $M \leq \|T\| \leq 2M$.

PROOF. Let $K(e, t)$ be a kernel satisfying the given conditions. Consider the integral $\int_R K(e, t)x(t)d\phi$ where $x \in L^p(R, \phi)$. By (a) the integral exists for each $e \in \mathcal{J}(S)$. Also by the Hölder inequality

$$\begin{aligned} \left| \int_R K(e, t)x(t)d\phi \right| &\leq \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'} \left\{ \int_R |x(t)|^p d\phi \right\}^{1/p} \\ &\leq M \|x\|. \end{aligned}$$

In particular $\sup_{e \in \mathcal{J}(S)} \left| \int_R K(e, t)x(t)d\phi \right| \leq \|x\| M$ and hence $\text{var}_{e \in \mathcal{J}(S)} \int_R K(e, t)x(t)d\phi \leq 2M \|x\|$. This shows that the function Tx is of bounded variation for each $x \in L^p(R, \phi)$. Also condition (b) implies that $\int_R K(e, t)x(t)d\phi$ is completely additive from $\mathcal{J}(S)$ to the reals. This shows that T is a bounded linear operator from $L^p(R, \phi)$ to $A(S, \gamma)$ with $\|T\| \leq 2M$.

To prove the necessity, assume that T is a bounded linear operator from $L^p(R, \phi)$ to $A(S, \gamma)$. If $Tx = F \in A(S, \gamma)$, fixing an $e \in \mathcal{J}(S, \gamma)$ will determine a linear functional f_e over $L^p(R, \phi)$ where $f_e(x) = F(e)$. By the Riesz representation theorem for linear functionals over $L^p(R, \phi)$, $f_e(x) = \int_R K(e, t)x(t)d\phi$ where $K(e, t)$ is in $L^{p'}(R, \phi)$. Hence $Tx = \int_R K(e, t)x(t)d\phi$. It remains to show that the kernel $K(e, t)$ satisfies the conditions of the theorem.

Condition (a) is evidently satisfied and so is (b) if $x(t)$ is taken as the characteristic function of the set E . To prove (c), let $\{E_i\}$ be any finite family of disjoint measurable subsets of R , each of finite nonzero measure. Let $e \in \mathcal{J}(S)$ and consider the sequence

$$\left\{ \frac{\int_{E_i} K(e, t)d\phi}{[\phi(E_i)]^{1/p}} \right\}.$$

By the lemma, if $\{a_i\}$ is any sequence of real numbers such that $\sum_{i=1}^{\infty} |a_i|^p \leq 1$, then $\sum_{i=1}^n |a_i|^p \leq 1$ for any n and the function $z(t) = \sum_{i=1}^n \{a_i \psi_{E_i}(t) / [\phi(E_i)]^{1/p}\}$ is in the unit sphere of $L^p(R, \phi)$.

Hence

$$\left| \int_R K(e, t) z(t) d\phi \right| = \left| \sum_{i=1}^n a_i \frac{\int_{E_i} K(e, t) d\phi}{[\phi(E_i)]^{1/p}} \right| \leq \|T\|.$$

If $\{a_i\}$ is as above, the sequence $\{\int_{E_i} K(e, t) d\phi / [\phi(E_i)]^{1/p}\}$ represents a linear functional over l^p of norm not exceeding $\|T\|$ and hence is in $l^{p'}$. This is true for any sequence $\{E_i\}$ satisfying the above conditions. Thus, using the expression for the norm in $(l^p)^* = l^{p'}$

$$\left\{ \sum_{i=1}^n \frac{\left| \int_{E_i} K(e, t) d\phi \right|^{p'}}{[\phi(E_i)]^{p'/p}} \right\}^{1/p'} = \left\{ \sum_{i=1}^n \frac{\left| \int_{E_i} K(e, t) d\phi \right|^{p'}}{[\phi(E_i)]^{p'-1}} \right\}^{1/p'} \leq \|T\|$$

for all finite sequences $\pi = \{E_i\}$ of measurable subsets of R of finite nonzero measure. Hence

$$\begin{aligned} \sup_{e \in \mathcal{J}(S)} \sup_{\pi} \left\{ \sum_{i=1}^n \frac{\left| \int_{E_i} K(e, t) d\phi \right|^{p'}}{[\phi(E_i)]^{p'-1}} \right\}^{1/p'} \\ = \sup_{e \in \mathcal{J}(S)} \left\{ p'\text{-var}_{E \in \mathcal{J}(R)} \int_E K(e, t) d\phi \right\} \\ = \sup_{e \in \mathcal{J}(S)} \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'} = M \leq \|T\|. \end{aligned}$$

Since $K(e, t)$ satisfies (a), (b), (c), $\|T\| \leq 2M$ by the first part of the theorem and $M \leq \|T\| \leq 2M$.

If $F(e)$ is a completely additive and absolutely continuous set function, there exists an essentially unique point function $x(t)$ such that $F(e) = \int_e x(t) d\phi$ by the Radon-Nikodym theorem. We shall denote this function by $x(t) = dF(e)/de$. If (R, ϕ) is a real interval under Lebesgue measure, this symbol represents ordinary differentiation.

COROLLARY 1. *A necessary and sufficient condition that T be a bounded linear operator from $L^p(R, \phi)$ to $L(S, \gamma)$ is that $Tx = d \int_R K(e, t) x(t) d\phi / de$ where the kernel $K(e, t)$ satisfies the conditions of the theorem and the norm is the same.*

COROLLARY 2. *A necessary and sufficient condition that T be a bounded linear operator from $L^p(R, \phi)$ to $L(S, \gamma)$ which takes essentially non-negative elements of $L^p(R, \phi)$ into essentially non-negative elements of $L(S, \gamma)$ is that the conditions of the theorem be fulfilled and in addi-*

tion, for each $e \in \mathcal{F}(S)$, $K(e, t) \geq 0$ for almost all $t \in R$. The norm of T in this case is given by

$$\|T\| = \sup_{e \in \mathcal{F}(S)} \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'}.$$

PROOF. It is evident that if for each e , $K(e, t) \geq 0$ for almost all t , the operator will be non-negative. Also if T is non-negative, then for each $e \in \mathcal{F}(S)$, $K(e, t) \geq 0$ for almost all $t \in R$, since for any $E \in \mathcal{F}(R)$, $\int K(e, t) \psi_E(t) d\phi = \int_E K(e, t) d\phi \geq 0$ if and only if $K(e, t) \geq 0$ for almost all t .

To prove the equality for the norm it was shown in the theorem that

$$\sup_{e \in \mathcal{F}(S)} \int_R K(e, t) x(t) d\phi \leq \|x\| \cdot \sup_{e \in \mathcal{F}(S)} \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'}$$

for any $x \in L^p(R, \phi)$. If $x(t) \geq 0$ almost everywhere and if $K(e, t)$ is as shown above, $\int_R K(e, t) x(t) d\phi \geq 0$ for each $e \in \mathcal{F}(S)$,

$$\text{var} \int_R K(e, t) x(t) d\phi = \sup_{e \in \mathcal{F}(S)} \int_R K(e, t) x(t) d\phi.$$

Now let $x \in L^p(R, \phi)$, $\|x\| = 1$. Then $x'(t) = |x(t)| \in L^p(R, \phi)$ and has unit norm. Also, if T is non-negative, $\|Tx\| = \|Tx'\|$ since if $x \in L^p(R, \phi)$ let $x^+(t) = x(t)$ if $x(t) \geq 0$ and 0 elsewhere, $x^-(t) = -x(t)$ if $x(t) < 0$ and 0 elsewhere. Then x^+ and x^- are both in $L^p(R, \phi)$ and are both non-negative. Then $Tx = T(x^+ - x^-) = Tx^+ - Tx^-$, $Tx' = Tx^+ + Tx^-$. If $y = Tx$, $y' = Tx'$, $|y(s)| \leq |y'(s)|$ for almost all s . Hence $\|y\| \leq \|y'\|$ in $L(S, \gamma)$. Thus if $\|x\| = 1$ there is a non-negative x' such that $\|x'\| = 1$ and $\|Tx\| \leq \|Tx'\|$. Thus $\|T\| = \sup \|Tx'\|$ where the upper bound is taken over all non-negative x' of unit norm. Hence for non-negative operators

$$\begin{aligned} \|T\| &= \sup_{x \geq 0, \|x\|=1} \sup_{e \in \mathcal{F}(S)} \int_R K(e, t) x(t) d\phi \\ &\leq \sup_{e \in \mathcal{F}(S)} \left\{ \int_R |K(e, t)|^{p'} d\phi \right\}^{1/p'} = M. \end{aligned}$$

Since $\|T\| \geq M$ for any operator, $\|T\| = M$.

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A NOTE ON CORRELATION FUNCTIONS AND STABILITY IN DYNAMICAL SYSTEMS¹

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1. In the system of differential equations

$$(1) \quad x' = f(x),$$

in which $x = (x_1, \dots, x_n)$ and $f(x) = (f_1, \dots, f_n)$ is of class C^1 , suppose that

$$(2) \quad \operatorname{div} f = \partial f_1 / \partial x_1 + \dots + \partial f_n / \partial x_n = 0$$

throughout a compact, invariant space, Ω , of (finite) positive v -measure. It will be supposed that Ω is the closure of an open set and that v refers to the ordinary n -dimensional Lebesgue volume measure in the x -space consisting of points $P = x$.

If $g(t)$ denotes any function of class (L) on every finite t -interval of $-\infty < t < \infty$, then $M_t[g(t)]$ will be defined as

$$M_t[g(t)] = \lim (2T)^{-1} \int_{-T}^T g(t) dt, \quad T \rightarrow \infty,$$

in case the limit exists. In view of the measure-preserving assumption (2), it follows from the Birkhoff ergodic theorem that if $f = f(P)$ denotes any function of class (L^2) on Ω , then the correlation function

$$(3) \quad c_P(s) = M_t[f(P_{t+s})\bar{f}(P_t)]$$

(which depends on f) exists as a continuous function of s on $-\infty < s < \infty$ for almost all points P of Ω , where $P = x(0)$ and $P_t = x(t)$ denotes

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