PARTIAL ORDER AND INDECOMPOSABILITY[1]

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In this note it is shown that, under suitable auxiliary hypotheses, no nontrivial "continuous" partial order can exist on an indecomposable continuum.

Let X be a regular T_1 space and let $\leqq$ be a binary relation on X. It is assumed that

(i) $\leqq$ is *transitive* and *reflexive*.

(ii) For each $x, y \in X$ there is a $z \in X$ *such that*

$$z \leqq x \quad and \quad z \leqq y.$$

The topology of X and the relation $\leqq$ are assumed to satisfy

(iii) *If* $x \in X$ *and if* U *is an open set about*

$$L(x) = \{y \mid y \leqq x\},$$

then there is an open set V *about* x *such that*

$$x' \in V \text{ implies } L(x') \subset U.$$

A discussion of this kind of "continuity" will be found in [**3**]. We asume also

(iv) *For each* $x \in X$ *the set* $L(x)$ *is compact and connected.*

THEOREM. *If* X *is a connected indecomposable space and if* $\leqq$ *satisfies the above conditions, then* $x \leqq y$ *for each* $x, y \in X$.

PROOF. If $x_1, x_2, \cdots, x_n \in X$, then (i) and (ii) imply that

$$L(x) \subset L(x_1) \cap \cdots \cap L(x_n)$$

for some $x \in X$. Hence

$$A = \bigcap \{L(x) \mid x \in X\}$$

is nonvoid, using (iv). If $a \in A$, then $L(a) \subset L(x)$ for each $x \in X$ by (i). It follows that $L(a) = A$ so that A is connected.

Suppose that the conclusion is false. Then $L(z) \neq X$ for some $z \in X$. Since $L(z)$ is closed there is a non-null open set W such that $L(z) \cap W^* = \square$, because X is regular. We use * for closure and $\backslash$ for complement. Let

Presented to the Society, June 19, 1954; received by the editors February 15, 1954.

[1] This work was done under Contract N7-onr-434, Task Order III, Navy Department, Office of Naval Research.

$$B = \{x \mid L(x) \subset X\backslash W^*\}.$$

The set $B \neq \square$ since $A \subset L(z)$ and $L(a) = A$ if $a \in A$. Also B is open because of (iii). We assert that

$$B = \bigcup \{L(y) \mid y \in B\}.$$

For if $x \in B$, then $x \in L(x)$ by (i). If $y \in B$, then $L(y) \subset X\backslash W^*$ and $x \in L(y)$ implies $L(x) \subset L(y)$ by (i) so that $L(x) \subset X\backslash W^*$ and $x \in B$. This representation shows that B is connected because it is the union of a family of connected sets all meeting the connected set $A \subset B$. Hence X contains the nonvoid open connected subset B with $B^* \neq X$. Thus X is not indecomposable. The proof is complete.

To obtain the initial assertion of our note let X be a continuum (compact connected Hausdorff space) and let

$$R = \{(x_1, x_2) \mid x_1 \leqq x_2\}.$$

If R is *closed* then it is quite easy to see that (iii) holds and that $L(x)$ is closed for each $x \in X$, e.g., [2], [3], or [5]. Hence the theorem obtains if R is closed and if $L(x)$ is connected for each $x \in X$. Of course we assume (i) and (ii).

Although this note is self-contained we refer to [2], [3], [4], and [5] for further results on "continuous partial orders." An interesting and formally analogous relation between "transitivity" and indecomposability has been given by Kuratowski [1, p. 147].

Bibliography

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