ITERATED LIMITS AND THE CENTRAL LIMIT THEOREM FOR DEPENDENT VARIABLES¹

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1. Introduction. §2 of this paper gives some results on iterated limits which may be considered generalizations of well-known results [1, p. 254]. §3 applies these results to give easy proofs of some central limit theorems for m -dependent variables. Compare Theorem 2 in [2] and Theorem 1 in [3].

2. Iterated probability limits. Here, we use the strong sense of an iterated limit: for constants a_{ij} , $i, j = 1, 2, \dots$, $\lim_j \lim_i a_{ij} = a$ means

$$(1) \quad \lim_{j \rightarrow \infty} \left(\limsup_{i \rightarrow \infty} |a_{ij} - a| \right) = 0.$$

We note that (1) holds if, and only if, for each $\epsilon > 0$ there exist integers $M, N_1, N_2, \dots$ such that if the pair (i, j) satisfies $j > M, i > N_j$, then $|a_{ij} - a| < \epsilon$.

DEFINITION 1. Let f, f_{ij} , $i, j = 1, 2, \dots$, be random variables. Then

$$\text{plim}_j \text{plim}_i f_{ij} = f$$

means, for every $\epsilon > 0$,

$$\lim_j \lim_i P(|f_{ij} - f| > \epsilon) = 0.$$

THEOREM 1. Let h_{ij}, g_{ij} , $i, j = 1, 2, \dots$ be random variables. Let G be a function such that at each of its continuity points x

$$\lim_j \lim_i P(g_{ij} \leq x) = G(x),$$

and suppose

$$\text{plim}_j \text{plim}_i h_{ij} = 0.$$

Then

$$\lim_j \lim_i P(g_{ij} + h_{ij} \leq x) = G(x).$$

Let $\epsilon = 8\delta > 0$ and a continuity point x of G be given. We shall exhibit integers $M, N_1, N_2, \dots$ such that

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$$(2) \quad |P(g_{ij} + h_{ij} \leq x) - G(x)| < \epsilon$$

if $j > M$ and $i > N_j$. First, choose β so that G is continuous at $x + \beta$, at $x - \beta$, and so that

$$(3) \quad |G(x + \beta) - G(x - \beta)| < \delta.$$

Then choose $M, N_1, N_2, \dots$ so that, simultaneously,

$$(4) \quad P(|h_{ij}| > \beta) < \delta,$$

$$(5) \quad |P(g_{ij} \leq x) - G(x)| < \delta,$$

$$(6) \quad |P(g_{ij} \leq x - \beta) - G(x - \beta)| < \delta,$$

$$(7) \quad |P(g_{ij} \leq x + \beta) - G(x + \beta)| < \delta$$

whenever (i, j) satisfies $j > M$ and $i > N_j$. Then for such a pair (i, j) , let $F(x) = P(g_{ij} + h_{ij} \leq x)$, $H(x, \beta) = P(g_{ij} + h_{ij} \leq x, |h_{ij}| \leq \beta)$, $L(x, \beta) = P(g_{ij} \leq x, |h_{ij}| \leq \beta)$, and $Q(x) = P(g_{ij} \leq x)$. We have

$$\begin{aligned} |F(x) - G(x)| &\leq |F(x) - H(x, \beta)| + |H(x, \beta) - L(x, \beta)| \\ &\quad + |L(x, \beta) - Q(x)| + |Q(x) - G(x)|. \end{aligned}$$

Now by (4) and (5), each of the terms on the right except the second is bounded by δ , and since $L(x - \beta, \beta) \leq H(x, \beta) \leq L(x + \beta, \beta)$ and $L(x - \beta, \beta) \leq L(x, \beta) \leq L(x + \beta, \beta)$,

$$|H(x, \beta) - L(x, \beta)| \leq |L(x + \beta, \beta) - L(x - \beta, \beta)| < 5\delta$$

by (4), (7), (3), and (6), since

$$\begin{aligned} &|L(x + \beta, \beta) - L(x - \beta, \beta)| \\ &\leq |L(x + \beta, \beta) - Q(x + \beta)| + |Q(x + \beta) - G(x + \beta)| \\ &\quad + |G(x + \beta) - G(x - \beta)| + |G(x - \beta) - Q(x - \beta)| \\ &\quad + |Q(x - \beta) - L(x - \beta, \beta)|. \end{aligned}$$

Hence

$$|F(x) - G(x)| < 8\delta = \epsilon,$$

which is condition (2).

THEOREM 2. *Under the conditions of Theorem 1, if there exist constants a_{ij} such that $\lim_j \lim_i a_{ij} = a > 0$, and if G is continuous at x/a , then $\lim_j \lim_i P(a_{ij} g_{ij} \leq x) = G(x/a)$.*

Using the artifice, for suitable i, j, γ

$$\begin{aligned}
& \left| P\left(g_{ij} \leq \frac{x}{a_{ij}}\right) - G\left(\frac{x}{a}\right) \right| \leq \left| P\left(g_{ij} \leq \frac{x}{a_{ij}}\right) - P\left(g_{ij} \leq \frac{x}{a}\right) \right| \\
& \quad + \left| P\left(g_{ij} \leq \frac{x}{a}\right) - G\left(\frac{x}{a}\right) \right|, \\
& \left| P\left(g_{ij} \leq \frac{x}{a_{ij}}\right) - P\left(g_{ij} \leq \frac{x}{a}\right) \right| \\
& \quad \leq \left| P\left(g_{ij} \leq \frac{x}{a-\gamma}\right) - P\left(g_{ij} \leq \frac{x}{a+\gamma}\right) \right|,
\end{aligned}$$

the proof is routine. The details are omitted.

3. Applications to partitioned sequences of m -dependent random variables. Let $x_1, x_2, \dots$ be an m -dependent sequence of random variables with zero means. For each pair (n, k) with $2m < k \leq n$, define

$$\begin{aligned}
y_i &= x_{i-k+1} + \dots + x_{i-m}, & i &= 1, 2, \dots, \\
g_{nk} &= \sum_1^{[n/k]} y_i, & t_{nk}^2 &= E(g_{nk}^2), \\
s_n^2 &= E(x_1 + \dots + x_n)^2, & h_{nk} &= \frac{1}{s_n} \left(\sum_1^n x_i - g_{nk} \right).
\end{aligned}$$

Since we shall be dealing with $\lim_k \lim_n$ relations, g_{nk} , $n < k$, may be defined indifferently.

According to Theorems 1 and 2, if

$$(8) \quad \lim_k \lim_n \frac{t_{nk}}{s_n} = 1,$$

$$(9) \quad \text{plim}_k \text{plim}_n h_{nk} = 0,$$

$$(10) \quad \lim_k \lim_n P\left(\frac{g_{nk}}{t_{nk}} \leq x\right) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^x e^{-t^2/2} dt,$$

then

$$(11) \quad \lim_n P\left(\frac{x_1 + \dots + x_n}{s_n} \leq x\right) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^x e^{-t^2/2} dt.$$

The following theorems give conditions which imply (8), (9), and (10).

THEOREM 3. *If there exist constants $\alpha > 2$, $B > 0$ such that*

$$(12) \quad n/s_n^2 < B, \quad n = 1, 2, \dots,$$

$$(13) \quad E(x_n^2) < B, \quad n = 1, 2, \dots,$$

$$(14) \quad \lim_n \frac{\left(\sum_1^n E(|x_i|^\alpha) \right)^{1/\alpha}}{s_n} = 0,$$

then condition (11) holds.

We first establish (8) and (9). One readily finds, for $2m < k \leq n$,

$$(15) \quad |s_n^2 - t_{nk}^2| < \left(\left[\frac{n}{k} \right] + k^2 \right) 8m^2 B$$

and

$$(16) \quad E(t_{nk}^2) < \frac{1}{s_n^2} \left(\left[\frac{n}{k} \right] + k^2 \right) 8m^2 B.$$

But, using (12),

$$(17) \quad \begin{aligned} \lim_k \lim_n \frac{[n/k] + k^2}{s_n^2} &= \lim_k \lim_n \frac{[n/k]}{s_n^2} \\ &= \lim_k \left(\frac{1}{k} \limsup_n \frac{n}{s_n^2} \right) = 0. \end{aligned}$$

Relations (15), (16), and (17) imply (8) and (9).

Condition (10) will be true, by Liapounoff's Theorem [4, p. 284], if, for large k ,

$$\lim_n \frac{\left(\sum_{i=1}^{[n/k]} E(|y_i|^\alpha) \right)^{1/\alpha}}{t_{nk}} = 0.$$

Now

$$E(|y_i|^\alpha) \leq k^\alpha \sum_{j=ik-k+1}^{ik-m} E(|x_j|^\alpha),$$

so that

$$\lim_n \frac{\left(\sum_{i=1}^{[n/k]} E(|y_i|^\alpha) \right)^{1/\alpha}}{t_{nk}} \leq \lim_n \frac{k \left(\sum_{i=1}^n E(|x_i|^\alpha) \right)^{1/\alpha}}{s_n} \cdot \frac{s_n}{t_{nk}} = 0$$

by (14) and (8), if k is large.

THEOREM 4. *If $x_1, x_2, \dots$ is a stationary m -dependent sequence with zero means and finite variances, then (11) holds.*

For in that case, (12) holds, and, since the variances are bounded, (8) and (9) are established as above. (10) holds, since, for each $k > 2m$, the sequence $y_1, y_2, \dots$ is stationary and independent.

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