LEGENDRE TRANSFORMATIONS OF CURVES

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In a recent paper of Wintner [1], an extension is made of a classical theorem on the Legendre transformation of a convex function (for details of the proof, see [3], and for related results, cf. [4; 5]). His assumptions are that the function be strictly convex and of class C^1 . Here we shall prove a more general result which eliminates both of these restrictions and shows that, in a sense, they are dual to one another. As two applications we mention a result of Kamke [2] on the Clairaut differential equation and a theorem on parallel curves (cf. [3, pp. 481-482]).

Let $y(x)$ be a convex bounded function of x on the interval (a, b) . Then y has monotone nondecreasing right- and left-hand derivatives $y'_-(x) \leq y'_+(x)$ in (a, b) . There will, in general, be two exceptional sets to consider in reference to these derivatives, the set j consisting of points x where $y'_-(x) \neq y'_+(x)$ and the set k of the closures of the x -intervals where $y'_+(x)$ (or $y'_-(x)$) is constant. j is an at most denumerable set of points while k is an at most denumerable set of closed intervals. Let K be the set of closures of intervals of y' -values satisfying $y'_-(x) < y' < y'_+(x)$ for some x on j , and J the set of y' -values for which $y' = y'_+(x)$ for some x on k . Furthermore, let $A = y'_+(a)$, $B = y'_-(b)$.

We now consider the correspondence $X = y'(x)$. This obviously assigns to every x not in j a unique X not in K and to every X not in J a unique x not in k . Thus $X = y'(x)$ is a unique (monotone and continuous) correspondence of $(a, b) - (j + k)$ onto $(A, B) - (J + K)$. In order to extend the domain of definition of this correspondence, put $\bar{x} = \bar{x}(X) = \text{l.u.b. } T(X)$, where $T(X)$ is the set of those x -values at which $y'_-(x) \leq X$. $\bar{x}(X)$ is in $T(X)$ and $\bar{x}(X)$ is continuous from the right.

We now define $Y = Y(X)$ as

$$(1) \quad Y = \bar{x}(X) \cdot X - y(\bar{x}(X)).$$

Letting ' after Y denote differentiation with respect to X we have the following

THEOREM. Y'_+ and Y'_- exist everywhere and are nondecreasing with $Y'_- \leq Y'_+$ and $Y'_+(X) = \bar{x}(X)$. Furthermore, $Y'_- \neq Y'_+$ only on J and Y'_+ constant only on intervals of K .

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In proving this theorem we make use of the following extension of the mean value theorem:

LEMMA. If $f(t)$ has in (t_1, t_2) a right- and left-hand derivate at every point and if $t' < t''$ are points of (t_1, t_2) , then there exists a t^* such that $t' < t^* < t''$ and $(t'' - t')f'_-(t^*) \leq f(t'') - f(t') \leq (t'' - t')f'_+(t^*)$ or $(t'' - t')f'_+(t^*) \leq f(t'') - f(t') \leq (t'' - t')f'_-(t^*)$.

To see this set

$$g(t) = f(t) - (t - t') \frac{f(t'') - f(t')}{t'' - t'}.$$

$g(t)$ has right- and left-hand derivates in (t', t'') and is certainly continuous on the closure of this interval. Furthermore $g(t'') = g(t') = f(t')$ and therefore $g(t)$ assumes a maximum or a minimum in the interior of the interval, say at $t = t^*$. Then $g'_+(t^*) \leq 0$, $g'_-(t^*) \geq 0$ or $g'_-(t^*) \leq 0$, $g'_+(t^*) \geq 0$ according as t^* is a maximum or a minimum. This is the desired result.

We shall now show that $Y_+(X)$ exists everywhere and is equal to $\bar{x}(X)$. Let X_1 be greater than X and $\Delta X = X_1 - X$, $\Delta \bar{x} = \bar{x}(X_1) - \bar{x}(X)$, etc. Then

$$\Delta Y = \bar{x}_1 \Delta X + X \Delta \bar{x} - \Delta y(\bar{x}).$$

By the previous lemma, there exists a x^* such that $\bar{x} < x^* < \bar{x}_1$ (unless $\bar{x}_1 = \bar{x}$ in which case $\Delta Y = \bar{x} \Delta X$) and

$$(\Delta \bar{x})y'_-(x^*) \leq \Delta y \leq (\Delta \bar{x})y'_+(x^*),$$

so that

$$\Delta Y \leq \bar{x}_1 \Delta X + X(\Delta \bar{x}) - (\Delta \bar{x})y'_-(x^*).$$

But $X \leq y'_-(x^*) \leq X_1$, hence

$$(2) \quad \Delta Y \leq \bar{x}_1 \Delta X.$$

Similarly,

$$\Delta Y = X_1 \Delta \bar{x} + \bar{x} \Delta X - \Delta y(\bar{x})$$

and, since x^* is strictly in between $\bar{x}$ and $\bar{x}_1$ (unless $\bar{x} = \bar{x}_1$ in which case $\Delta Y = \bar{x} \Delta X$), $X \leq y'_+(x^*) \leq X_1$. Thus

$$(3) \quad Y \geq \bar{x} \Delta X.$$

We therefore have, by (2) and (3),

$$\bar{x} \leq \Delta Y / \Delta X \leq \bar{x}_1.$$

Letting X_1 tend to X we get, since $\bar{x}$ is continuous from the right, $Y_+(X) = \bar{x}(X)$.

Since $\bar{x}$ is a nondecreasing function of X , Y_+ is nondecreasing. Furthermore, $Y_-(X) = Y_+(X-0) = \bar{x}(X-0)$, $Y_-(X) \leq Y_+(X)$, and $Y_-(X)$ is nondecreasing. The points where $Y_-(X) \neq Y_+(X)$ are those where $\bar{x}(X-0) \neq \bar{x}(X)$ or where $y'_-(x)$ is constant on some interval to the left of $\bar{x}$, thus belonging to J . Similarly $Y_+ = \bar{x}$ is constant only on intervals of K .

Y is thus a convex function of X . We can, therefore, apply the preceding transformation (1) on $Y = Y(X)$. We thus put $x = Y(X)$ on $[A, B] - (J+K)$ and $\bar{X} = \bar{X}(x) = \text{l.u.b. } t(x)$ where $t(x)$ is the set of X such that $Y_-(X) \leq x$.

Finally, let $y^* = y^*(x)$ be defined by

$$(4) \quad \begin{aligned} y^* &= \bar{X}(x) \cdot x - Y[\bar{X}(x)] \\ &= \bar{X}(x) \cdot x - \bar{x}[\bar{X}(x)]\bar{X} + y(\bar{x}[\bar{X}(x)]). \end{aligned}$$

Now, by the definitions of the functions $\bar{x}(X)$ and $\bar{X}(x)$, $\bar{x}[\bar{X}(x_0)]$ is the largest x such that $y'_-(x) \leq y'_-(x_0)$. For x not in k , $\bar{x}[\bar{X}(x)] = x$, in which case

$$y^* = \bar{X} \cdot x - x \cdot \bar{X} + y(x) = y.$$

For x in an interval of k , $y(x)$ is linear with the constant slope $\bar{X}$ and assumes the value $y(\bar{x}[\bar{X}(\bar{x})])$ at $x = \bar{x}$, and so the last part of (4) shows that $y^* = y$.

Hence $y^* \equiv y$ for all x in $[a, b]$. Thus the transformation (1) treated above is involutory with

$$Y = Y(X) = \bar{x}X - y(\bar{x}), \quad Y_+(X) = \bar{x}$$

and

$$y = y(x) = \bar{X}x - Y(\bar{X}), \quad y'_+(x) = \bar{X}.$$

If y has a derivative everywhere, then j is empty and therefore K is empty. Thus if y is differentiable everywhere, Y is strictly convex and vice versa. Similarly, if y is strictly convex, the Y has a derivative everywhere and vice versa. Finally, if y or Y are strictly convex and have derivatives everywhere, or if both are strictly convex or both are everywhere differentiable, then both are strictly convex and everywhere differentiable.

As a first application, we mention the following theorem:

The (outside) parallel curves of a convex curve are of class C^1 (at least).

In effect, rolling a circle outside the curve increases the support

function which means adding a strictly convex term to the Legendre transform (for suitable choice of coordinates).

As a second application, we mention the result of Kamke [2] on the Clairaut differential equation, namely, the existence of a non-linear solution of $y = y'x + f(y')$ when f is strictly convex. Here, the Legendre transform of f provides a solution if it is differentiable, which it must be in virtue of the strict convexity of f .

REFERENCES

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