A REMARK ON A NORM INEQUALITY FOR SQUARE MATRICES

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In a recent paper W. Gautschi [1] obtained the following inequality for a non-nilpotent n-square matrix A:

$$(1) \qquad \left(\sum_{j=1}^{n} |\lambda_j|^{2p}\right)^{1/2} \leqq \|A^p\| \leqq c_0 p^{k-1}\left(\sum_{j=1}^{n} |\lambda_j|^{2p}\right)^{1/2}$$

where $\|A\|^2 = \sum_{i,j=1}^{n} |a_{ij}|^2$, p is an integer, c_0 a constant depending only on A, and k the maximum multiplicity of any characteristic root of A. By a reduction to Jordan canonical form an inequality of type (1) may be obtained for any convergent matrix power series $g(A)$, $g(z) = \sum_{n=0}^{\infty} a_n z^n$, a_n real, which in the case $g(z) = z^p$ implies (1).

THEOREM. *Let $g(z)$ have radius of convergence $\rho > 0$. Let A be an n-square matrix whose characteristic roots satisfy $|\lambda_i| < \rho$. Let the multiplicity of λ_i be k_i, $i = 1, \cdots, \alpha$, and $k = \max k_i$. Then if A is not nilpotent there exists a constant c depending only on A such that*

$$\left(\sum_{j=1}^{n} | g(\lambda_j) |^2\right)^{1/2} \leqq \|g(A)\| \leqq c\left(\sum_{j=1}^{\alpha} \sum_{s=0}^{k_j-1} \frac{(k_j - s)}{(s!)^2} | g^{(s)}(\lambda_j) |^2\right)^{1/2}.$$

PROOF. A very slight extension of Gautschi's argument yields the lower bound. For by Schur's theorem [2] there exists a unitary matrix U reducing A to triangular form D with characteristic roots along the main diagonal. Then

$$UAU^* = D \quad \text{and} \quad UU^* = I$$

imply

$$\begin{aligned}
\|g(A)\|^2 &= \operatorname{tr}(g(A)g^*(A)) = \operatorname{tr}(g(A)g(A^*)) \\
&= \operatorname{tr}(Ug(A)g(A^*)U^*) = \operatorname{tr}(Ug(A)U^*Ug(A^*)U^*) \\
&= \operatorname{tr}(g(UAU^*)g(UA^*U^*)) = \operatorname{tr}(g(D)g(D^*)) \\
&= \operatorname{tr}(g(D)g^*(D)) \geqq \sum_{j=1}^{n} | g(\lambda_j) |^2
\end{aligned}$$

since $g(D)$ is a triangular matrix with $g(\lambda_j)$ along the main diagonal. To obtain the upper bound let T be such a nonsingular matrix that $TAT^{-1} = \operatorname{diag}(E_1, \cdots, E_\alpha) = \operatorname{diag} E_j$; $E_j = \lambda_j I_j + U_j$, I_j the k_j-square

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identity matrix and U_j the k_j-square matrix with 1 along the superdiagonal and 0 elsewhere. Then setting $c=\|T\|\|T^{-1}\|$, we have

$$\|g(A)\| = \|T^{-1}Tg(A)T^{-1}T\| = \|T^{-1}g(TAT^{-1})T\| \leqq c\|g(\text{diag } E_j)\| \\ = c\|\text{diag } g(E_j)\|.$$

An elementary calculation shows (S. Lefschetz [3])

$$g(E_j) = g(\lambda_j I_j + U_j) = \begin{pmatrix} g(\lambda_j) & g^{(1)}(\lambda_j) & \cdots & g^{(k_j-1)}(\lambda_j)/(k_j-1)! \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & g^{(1)}(\lambda_j) \\ 0 & \cdot & \cdot & g(\lambda_j) \end{pmatrix}$$

and hence by summing down the diagonals,

$$\|g(E_j)\|^2 = \sum_{s=0}^{k_j-1} ((k_j - s)/(s!)^2) \mid g^{(s)}(\lambda_j) \mid^2$$

and

$$\|g(A)\|^2 = c^2 \sum_{j=1}^{\alpha} \|g(E_j)\|^2 \leqq c^2 \sum_{j=1}^{\alpha} \sum_{s=0}^{k_j-1} \frac{(k_j - s)}{(s!)^2} \mid g^{(s)}(\lambda_j) \mid^2,$$

and the proof is complete.

If $g(z)=z^p$ we may easily estimate $\|g(E_j)\|$. We distinguish two possibilities: (i) $p>k_j-1$, (ii) $p\leqq k_j-1$. If (i), $\lambda_j=0$ implies $\|g(E_j)\|=0$, and $\lambda_j\neq 0$ implies

$$\|g(E_j)\|^2 = \mid \lambda_j \mid^{2p} \sum_{s=0}^{k_j-1} \binom{p}{s}^2 (k_j - s) \mid \lambda_j \mid^{-2s} \leqq c_1 \mid \lambda_j \mid^{2p} p^{2k-2}.$$

If (ii), $\lambda_j=0$ implies $\|g(E_j)\|^2=(k_j-p)(p!)^2\leqq c_2 p^{2k-2}$, and again $\lambda_j\neq 0$ implies the same conclusion as in (i), where c_1 and c_2 depend only on the multiplicities and magnitudes of the λ_j. Thus

$$\|g(A)\|^2 \leqq c^2 p^{2k-2} \left(c_2 + c_1 \sum_{j=1}^{n} \mid \lambda_j \mid^{2p} \right) \leqq c_0^2 p^{2k-2} \sum_{j=1}^{n} \mid \lambda_j \mid^{2p}$$

since A not nilpotent is equivalent to $\sum_{j=1}^n |\lambda_j|^2 \neq 0$. Noting that $\sum_{j=1}^n |g(\lambda_j)|^2 = \sum_{j=1}^n |\lambda_j|^{2p}$, Gautschi's result follows. We may employ the inequality to unify conveniently the familiar results concerning the asymptotic behavior of the system of ordinary differential equations $\dot{x}=Ax$, A an n-square constant matrix, x an n-column vector, with fundamental matrix of solutions exp (tA). Consider $g(z)=\exp(tz)$, $g^{(s)}(z)=t^s \exp(tz)$, $t\geqq 0$, and the resulting inequality

$$\left(\sum_{j=1}^{n}|\exp(t\lambda_j)|^2\right)^{1/2} \leqq \|\exp(tA)\|$$

$$\leqq c\left(\sum_{j=1}^{\alpha}\sum_{s=0}^{k_j-1}\frac{(k_j-s)}{(s!)^2}|(t^s\exp(t\lambda_j))|^2\right)^{1/2}.$$

Letting $r_j = \mathrm{Re}\,(\lambda_j)$ we have

$$\left(\sum_{j=1}^{n}\exp(2r_jt)\right)^{1/2} \leqq \|\exp(tA)\|$$

$$\leqq c\left(\sum_{j=1}^{\alpha}\sum_{s=0}^{kj-1}\frac{(k_j-s)}{(s!)^2}t^{2s}\exp(2tr_j)\right)^{1/2}$$

and we conclude immediately that:

(i) $r_j>0$ for some j implies $\lim_{t\to\infty}\|\exp(tA)\|=\infty$,

(ii) $r_j<0$ for all j implies $\lim_{t\to\infty}\|\exp(tA)\|=0$,

(iii) $r_j\leqq 0$ for all j and $r_j=0$ only if $k_j=1$ implies $\|\exp(tA)\|$ is bounded on $[0, \infty)$.

In case A is nilpotent, $\exp(tA)$ has as entries polynomials in t.

References

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3. S. Lefschetz, *Lectures on differential equations*, Annals of Mathematics Studies, no. 14, p. 6.

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