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MULTIPLICATIVE GROUPS OF ANALYTIC FUNCTIONS

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Let D be a *proper* subdomain of the Riemann sphere, and let $M(D)$ be the multiplicative group of all regular single-valued analytic functions on D which have no zeros in D. It is known [1] that the algebraic structure of the ring $R(D)$ of all regular single-valued analytic functions on D determines (and is determined by) the conformal type of D. In this paper we ask the question: what information about D does the algebraic structure of $M(D)$ give, and, conversely, which properties of D determine the algebraic structure of $M(D)$? The answer is, briefly, that *$M(D_1)$ and $M(D_2)$ are isomorphic if and only if D_1 and D_2 have the same connectivity.*

Here the connectivity of D is k if the complement of D has k components, and is ∞ if the complement of D has infinitely many (countable or power of the continuum) components. The structure of $M(D)$ is described in more detail in the theorem below.

If we associate with each $f\in M(D)$ the function $g=f/|f|$ we obtain a subgroup (isomorphic to $M(D)$) of the multiplicative group $C(D)$ of all continuous functions from D into the unit circumference. Such functions have been studied in great detail by Eilenberg [2]. It is worth noting that our theorem is valid if we replace $M(D)$ by $C(D)$, and that the proof is essentially the same; but it seems more interesting to stay within the smaller group.

Before stating the theorem, it is convenient to define two subgroups of $M(D)$.

(1) Fix a point $z_0\in D$ and let $G(D)$ be the set of all $f\in M(D)$ such that $f(z_0)=1$. Then $M(D)$ is the direct product of $G(D)$ and the multiplicative group of the nonzero complex numbers, and $G(D)$

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contains no elements of finite order. Without loss of generality we shall assume that $z_0=0$.

(2) Let $\Gamma(D)$ be the set of all $f\in G(D)$ for which the equation $g^n=f$ has a solution $g\in G(D)$, for every nonzero integer n. It is clear that, given f and n, there is only one such g, and that $g\in\Gamma(D)$.

THEOREM. (i) *The groups* $\Gamma(D_1)$ *and* $\Gamma(D_2)$ *are isomorphic for any two proper subdomains of the Riemann sphere.*

(ii) $G(D)$ *is (isomorphic to) the direct product of* $\Gamma(D)$ *and* $G(D)/\Gamma(D)$.

(iii) *If the complement of* D *has* $k+1$ *components, then* $G(D)/\Gamma(D)$ *is the direct product of* k *infinite cyclic groups.*

(iv) *If the complement of* D *has infinitely many components, then* $G(D)/\Gamma(D)$ *is isomorphic to the additive group of all integer-valued functions on a countable space.*

The proof will be broken up into several steps.

Step 1. *The structure of* $\Gamma(D)$. Since we consider single-valued functions only, and since every $f\in\Gamma(D)$ has nth roots of all orders, we see that $f\in\Gamma(D)$ if and only if the total change in the argument of $f(z)$, as z travels over any closed path in D, is zero. Hence $f\in\Gamma(D)$ if and only if f has a single-valued logarithm in D, i.e., if f is an exponential. Writing $f(z)=e^{g(z)}$, where g is normalized by $g(0)=0$, the one-to-one correspondence $f\leftrightarrow g$ is an isomorphism between $\Gamma(D)$ and the additive group $A(D)$ of all single-valued regular analytic functions g on D such that $g(0)=0$. Part (i) of the theorem will follow if we can show that $A(D_1)$ and $A(D_2)$ are isomorphic.

We are going to show a little more. Let $L(D)$ be the vector space (over the complex field) whose members are the members of $A(D)$; we shall see that $L(D_1)$ and $L(D_2)$ are isomorphic.

Let dim $L(D)$ stand for the cardinality of a Hamel basis of $L(D)$. It is enough to show that dim $L(D)$ does not depend on D. Let U be an open circular disc (the unit disc, without loss of generality) such that $U\subset D$, let Z be the finite plane, and assume (again without loss of generality) that $D\subset Z$. Then

$$L(U)\supset L(D)\supset L(Z)$$

so that

$$\dim L(U)\geqq \dim L(D)\geqq \dim L(Z).$$

To prove that the equality signs actually hold, we exhibit an isomorphism between $L(U)$ and a subspace of $L(Z)$: to $f\in L(U)$, $f(z)=\sum_1^\infty a_nz^n$, associate the function $g\in L(Z)$ given by $g(z)=\sum_1^\infty a_n(z/n)^n$.

Having proved part (i) of the theorem, we shall from now on write Γ in place of $\Gamma(D)$.

Step 2. *Domains of finite connectivity.* Suppose D is bounded, and let $C_1, \cdots, C_k$ be the bounded components of the complement of D. Choose points a_i $(i=1, \cdots, k)$ in C_i. For every $f \in G(D)$ there is a unique set of integers $n_1, \cdots, n_k$ and a unique function $g \in \Gamma$ such that

$$f(z) = g(z) \prod_{i=1}^{k} \left(1 - \frac{z}{a_i}\right)^{n_i}. \tag{*}$$

The integer n_j is obtained by considering the change in $\arg f(z)$ as z travels around a simple closed curve which contains C_j (and no other C_i) in its interior. The functions $(1-z/a_i)$ are the generators of the factor group $G(D)/\Gamma$.

The representation (*) proves the theorem completely, for domains of finite connectivity.

Step 3. *The structure of* $G(D)/\Gamma$ *for domains of infinite connectivity.* We shall assume that the complement C of D is bounded, and will construct a countable set of simple polygons P_t, with interiors Q_t, as follows. Q_1 contains C. P_{11} and P_{12} are in Q_1, are exterior to each other, $C \cap Q_{11} \neq 0$, $C \cap Q_{12} \neq 0$, and

$$C = (C \cap Q_{11}) \cup (C \cap Q_{12}).$$

We continue in this manner: if, for some t, Q_t contains only one component of C, we call P_t a *final* polygon. If P_t is not final, we construct P_{10t+1} and P_{10t+2} in Q_t, exterior to each other, such that $C \cap Q_{10t+1} \neq 0$, $C \cap Q_{10t+2} \neq 0$, and

$$C \cap Q_t = (C \cap Q_{10t+1}) \cup (C \cap Q_{10t+2}).$$

We may construct these polygons so that they satisfy one further condition: for every $\delta > 0$ there exists an integer t_0 such that Q_t contains no two components of C whose distance exceeds δ whenever $t > t_0$. (For the details of such constructions, we refer to [**3**, pp. 46–56].)

We let T be the set of all t such that P_t is a nonfinal polygon of the set so constructed. T is a countable set of integers (whose decimal representations consist of ones and twos), and is the space mentioned in part (iv) of the theorem.

Let T' be the set of all integers t for which P_t exists. For $t \in T'$ and $f \in G(D)$, put

$$2\pi w_f(t) = \Delta_t \arg f(z),$$

where Δ_t indicates the increment as z travels around P_t in the positive direction. Then $w_f(1)=0$, and for $t\in T$ the argument principle implies that

$$w_f(10t+1)+w_f(10t+2) = w_f(t).$$

For $f\in G(D)$, we now define an integer-valued function s_f on T by:

$$s_f(t) = w_f(10t+1) - w_f(t) = -w_f(10t+2).$$

Then $s_{fg}(t)=s_f(t)+s_g(t)$, so that the mapping $f\rightarrow s_f$ is a homomorphism of $G(D)$ into the additive group S of all integer-valued functions on T. Since $s_f(t)=s_g(t)$ for all $t\in T$ if and only if $w_f(t)=w_g(t)$ for all $t\in T'$, and this last equality occurs if and only if $f^{-1}g\in\Gamma$, we see that Γ is the kernel of the above homomorphism, so that $G(D)/\Gamma$ is isomorphic to a subgroup of S.

We next wish to show that $G(D)/\Gamma$ is isomorphic to S, i.e., the homomorphism of $G(D)$ into S is onto.

Let $s\in S$. We shall construct a function $f\in G(D)$ such that $s_f=s$. For $t\in T$, choose $x_t\in C\cap Q_{10t+1}$, $y_t\in C\cap Q_{10t+2}$, such that $|x_t-y_t|\rightarrow 0$ as $t\rightarrow\infty$ (this is possible, by the last condition which we imposed on the polygons). Putting

$$u_t(z) = (x_t - y_t)/(z - x_t) \qquad (t\in T,\ z\in D)$$

we have $1-u_t(z)=(z-y_t)/(z-x_t)$. Since $x_t-y_t\rightarrow 0$, there exists a sequence of positive integers k_t (for instance, $k_t=t+|s(t)|$) such that

$$\sum_{t\in T} s(t)\{u_t(z)\}^{k_t}$$

converges absolutely for $z\in D$, and uniformly in every closed subset of D. Like in the standard proof of the Weierstrass factorization theorem for entire functions we now see that the product

$$g(z) = \prod_{t\in T}\left\{[1-u_t(z)]\exp\left[u_t(z)+\cdots+\frac{[u_t(z)]^{k_t-1}}{k_t-1}\right]\right\}^{s(t)}$$

defines a regular analytic function with no zeros in D. Putting $f(z)=g(z)/g(0)$, we obtain the desired function: $s_f(t)=s(t)$ for all $t\in T$.

This completes the proof of part (iv) of the theorem.

Step 4. $G(D)$ *has* Γ *as a direct factor.* In the case of finite connectivity this was trivial; in the general case, we proceed as follows:

By Zorn's lemma, $G(D)$ contains a subgroup H which is maximal with respect to the two properties

(A) $\Gamma \cap H = \{1\}$ (the set consisting of the identity element alone);

(B) If $h \in G(D)$ and $h^n \in H$ for some integer $n \neq 0$, then $h \in H$.

By (A), the group ΓH generated by Γ and H is the direct product of Γ and H. Assume that $G(D)$ contains an element f which is not in ΓH. Let $H(f)$ be the set of all $w \in G(D)$ such that $w^m = f^n h$ for some $h \in H$ and some integers m, n ($m \neq 0$). Then $H(f)$ is a subgroup of $G(D)$ which contains H properly and which satisfies (B). Since H is maximal, $H(f)$ cannot satisfy (A), so that there exists an element $g_0 \neq 1$ in Γ, and an integer $m \neq 0$, such that $g_0^m = f^n h$. Since $g_0 \neq 1$, $g_0^m \neq 1$, so that $n \neq 0$ (by (A)). Choose $g \in \Gamma$ such that $g^n = g_0^m$; this is possible since Γ is closed with respect to the operation of taking roots. Then $(g^{-1}f)^n \in H$, (B) implies that $g^{-1}f \in H$, but this contradicts the assumption that $f \notin \Gamma H$.

Thus $G(D) = \Gamma H$. This establishes part (ii) and completes the proof of the theorem.

We conclude with the following remark. Since $G(D) = \Gamma H$ and $G(D)/\Gamma$ is isomorphic to S, we see that H and S are isomorphic. Moreover, this isomorphism is induced by the natural mapping $f \rightarrow s_f$ of H onto S. In Step 3 we constructed, for each $s \in S$, a function $f \in G(D)$ such that $s_f = s$; but the functions obtained by means of this construction do not form a group, and it seems difficult to modify the construction (by proper choice of convergence factors) so as to make the abstract considerations of Step 4 unnecessary.

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