

ON THE INTERSECTIONS OF THE COMPONENTS OF A DIFFERENCE POLYNOMIAL

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The purpose of this note is to prove the following theorem:

Solutions common to two distinct components¹ of the manifold of a difference polynomial annul the separants of the polynomial.

We begin by considering a field K , not necessarily a difference field, and a set of polynomials F_1, F_2, \dots, F_p in $K[u_1, \dots, u_q; x_1, \dots, x_p]$, the u_i and x_j being indeterminates, where for each $j, j=1, \dots, p-1, F_j$ is free of the $x_k, k>j$. We shall show that *any zero of F_1, \dots, F_p which annuls no formal partial derivative $\partial F_j/\partial x_j$ belongs to just one component of $\{F_1, \dots, F_p\}_0$.*² Furthermore, this component is of dimension q .

PROOF. Let $u_i = \gamma_i, i=1, \dots, q; x_j = \alpha_j, j=1, \dots, p$, be a zero of F_1, \dots, F_p which annuls no $\partial F_j/\partial x_j$. If $\gamma'_1, \dots, \gamma'_q; \alpha'_1, \dots, \alpha'_p$ is a zero of F_1, \dots, F_p which specializes to $\gamma_1, \dots, \gamma_q; \alpha_1, \dots, \alpha_p$, then this zero too annuls no $\partial F_j/\partial x_j$. It follows from this that α'_1 is algebraic over $K(\gamma'_1, \dots, \gamma'_q)$, and that for each $k, 1 < k \leq p, \alpha'_k$ is algebraic over $K(\gamma'_1, \dots, \gamma'_q; \alpha'_1, \dots, \alpha'_{k-1})$. This implies that a component of the manifold of $\{F_1, \dots, F_p\}_0$ containing $\gamma_1, \dots, \gamma_q; \alpha_1, \dots, \alpha_p$ is of dimension at most q .

We let $u_i = t_i + \gamma_i, i=1, \dots, q; x_j = \alpha_j + h_j, j=1, \dots, p$. Here the t_i denote new indeterminates and the h_j certain formal series in positive integral powers of the t_i . We shall show that the h_j may be so chosen that these substitutions annul F_1, \dots, F_p . In fact, the lemma proved in [3] shows that for each $k, 1 \leq k \leq p$, we may annul F_k by substitutions $u_i = t_i + \gamma_i, i=1, \dots, p, x_j = s_j + \alpha_j, j < k, x_k = \alpha_k + h'_k$, where the $s_j, j=1, \dots, p$, are new indeterminates, and h'_k is a formal series in positive integral powers of the t_i and $s_j, j < k$. For h_1 we take h'_1 ; for h_2 we take the result of replacing s_1 in h'_2 by h'_1 ; and so on.

With the h_j as described let Σ denote the set of polynomials in $K[u_1, \dots, u_q; x_1, \dots, x_p]$ which are annulled by the above substitutions. Evidently Σ is a prime p. i. (polynomial ideal). Its dimen-

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¹ The term "component," not previously defined for difference manifolds, is to have the expected meaning: a component is a maximal irreducible submanifold of a manifold. For definitions of other terms and symbols see [2; 3; 4].

² As in Chapter IV of [1] this notation indicates the perfect polynomial ideal generated by F_1, F_2, \dots, F_p .

sion is q and the u_i form a parametric set. For evidently Σ can contain no polynomial in the u_i alone, while the conclusion of the preceding paragraph but one shows that its dimension cannot exceed q . The result of that paragraph also shows that no component of $\{F_1, \dots, F_p\}_0$ can properly contain the manifold of Σ , for then its dimension would exceed q . Hence this manifold is itself a component of $\{F_1, \dots, F_p\}_0$.

Let \mathcal{M} be a component of $\{F_1, \dots, F_p\}_0$ which contains $\gamma_1, \dots, \gamma_q; \alpha_1, \dots, \alpha_p$, and let Λ be the prime p. i. in $K[u_1, \dots, u_q; x_1, \dots, x_p]$ whose manifold is \mathcal{M} . We must show that Λ is Σ . If Λ is of dimension 0 then, because Σ vanishes for a zero of Λ , and every zero must be a generic zero, Σ is contained in Λ . Since the manifolds of both are components of the same manifold, it follows that $\Lambda = \Sigma$ (and that $q = 0$). We suppose that Λ is of positive dimension, and that Λ and Σ are distinct. Then, since Λ cannot contain Σ , there is a polynomial P in Σ which is not in Λ . Then Λ possesses a zero not annulling P of the form

$$(1) \quad \begin{aligned} u_i &= \gamma_i + g_i, & i &= 1, \dots, q; \\ x_j &= \alpha_j + f_j, & j &= 1, \dots, p, \end{aligned}$$

where the g_i and the f_j are series in positive integral powers of a parameter t .

It is evident that (1) is a zero of F_1, \dots, F_p . We may also obtain a zero of these polynomials of the form

$$(2) \quad \begin{aligned} u_i &= \gamma_i + g_i, & i &= 1, \dots, q; \\ x_j &= \alpha_j + f'_j, & j &= 1, \dots, p, \end{aligned}$$

where the f'_j are again series in positive integral powers of t , and each f'_j is obtained by replacing the t_i , $i=1, \dots, p$, in h_j by the corresponding g_i . It is evident from the manner of formation of (2) that it is a zero of Σ .

We replace the u_i in F_1 by $\gamma_i + g_i$, $i=1, \dots, q$. There results a polynomial \bar{F}_1 in x_1 with coefficients power series in t . \bar{F}_1 vanishes, but its formal derivative $d\bar{F}_1/dx_1$ does not, when we put $t=0$, $x_1=\alpha_1$. It follows that there is a unique series f''_1 in positive integral powers of t such that $x_1=\alpha_1+f''_1$ is a solution of $\bar{F}_1=0$. We now replace the u_i , $i=1, \dots, q$, and x_1 in F_2 by $\gamma_i + g_i$ and $\alpha_1 + f''_1$ respectively to obtain a polynomial \bar{F}_2 in x_2 with coefficients power series in t . As before, we see that $\bar{F}_2=0$ possesses a solution $x_2=\alpha_2+f''_2$, where f''_2 is a series in positive integral powers of t . This series is unique. Continuing in this way we find uniquely determined f''_j , $j=1, \dots, p$, which are series in positive integral powers of t such that $u_i=\gamma_i+g_i$,

$i=1, \dots, q; x_j=\alpha_j+f_j'', j=1, \dots, p$, is a zero of F_1, \dots, F_p .

The uniqueness of the f_j'' shows that (1) and (2) are identical. Hence (1) annuls Σ , and, in particular, it annuls P . We have thus obtained a contradiction. This completes the proof of our statement concerning the zeros of F_1, \dots, F_p .

Now let \mathcal{F} be a difference field and A a polynomial of $\mathcal{F}\{y_1, \dots, y_n\}$. We shall prove the theorem stated at the beginning of this note. We may suppose that a transform of some y_i , say of y_n , appears effectively in A . Let $y_i=\alpha_i, i=1, \dots, n$, be a zero of A . It will suffice to assume that the α_i are not a zero of the y_n -separant of A and show that this implies that only one component of the manifold of A contains the α_i .

It is evident that the α_i must annul just one irreducible factor, say F , of A , and do not annul the y_n -separant of F . Hence we need merely show that the α_i are contained in only one component of the manifold of F . We shall suppose that this is not so and obtain a contradiction. We assume first that F is of equal order and effective order in y_n .

Let \mathcal{M}_1 and \mathcal{M}_2 denote two distinct components of the manifold of F , each containing the α_i . Let Σ_1 and Σ_2 denote the corresponding reflexive prime difference ideals. We denote by h the order of F in y_n . Since the α_i do not annul the y_n -separant of F , y_1, \dots, y_{n-1} constitute a parametric set for both Σ_1 and Σ_2 , and these ideals are both of order h in y_n .

We choose an integer m such that the first $m+1$ polynomials of a characteristic sequence of Σ_1 do not constitute the beginning of a characteristic sequence of Σ_2 . Let Σ_{1m} and Σ_{2m} denote the sets consisting of those polynomials of Σ_1 and Σ_2 respectively which involve the $y_n \hat{k}, 0 \leq k \leq m+h$, and a finite subset S of the $y_{ij}, i < n$. S is to include all those $y_{ij}, i < n$, which appear effectively, or whose transforms appear effectively, in F, F_1, \dots, F_m or in the first $m+1$ polynomials of a characteristic sequence of Σ_1 or in the first $m+1$ polynomials of a characteristic sequence of Σ_2 .

Σ_{1m} and Σ_{2m} may be regarded as prime p.i.'s in the ring $\mathcal{F}[S, y_{n0}, y_{n1}, \dots, y_{n,m+h}]$. The y_{ij} of S and the $y_{nk}, k < h$, constitute a parametric set for both Σ_{1m} and Σ_{2m} . Let s denote the number of indeterminates in this parametric set.

Our earlier result concerning polynomial ideals shows that there is a unique component \mathcal{M} of the manifold of $\{F, F_1, \dots, F_m\}_0$, regarded as an ideal of $\mathcal{F}[S, y_{n0}, y_{n1}, \dots, y_{n,m+h}]$, which contains the zero $y_{ij}=\alpha_{ij}$ of this ideal. The dimension of \mathcal{M} is s , for s corresponds to q of the earlier proof.

Now both Σ_{1m} and Σ_{2m} contain $\{F, F_1, \dots, F_m\}_0$, while both have

the zero $y_{ij} = \alpha_{ij}$. Hence their manifolds are in \mathcal{M} . Since their manifolds are of dimension s , however, they must coincide with \mathcal{M} . Hence Σ_{1m} and Σ_{2m} are identical. But m was chosen so that Σ_{1m} contains a polynomial which is not in Σ_{2m} , namely one of the first $m+1$ polynomials of a characteristic sequence of Σ_1 . We have obtained a contradiction. This completes the proof of the theorem in the case that F is of equal order and effective order in y_n .

If the order of F in y_n exceeds its effective order by $d > 0$, we replace each y_{nk} in F by z_{k-d} , where z is a new indeterminate, and subscripts attached to z denote transforming. F goes into an irreducible polynomial \bar{F} which is of equal order and effective order in z .

Evidently each component $\bar{\mathcal{M}}$ of the manifold of \bar{F} corresponds to a unique component \mathcal{M} of the manifold of F , and, conversely, each component of the manifold of F is obtained from a unique component of the manifold of \bar{F} . The correspondence may be described as follows: each solution in $\bar{\mathcal{M}}$ is obtained from a solution in \mathcal{M} by leaving unchanged the elements assigned as values to y_1, \dots, y_{n-1} , and assigning to y_n an element whose d th transform is the element assigned as the value of z in $\bar{\mathcal{M}}$. This correspondence carries solutions common to two components of the manifold of F into solutions common to two components of the manifold of \bar{F} . Solutions annulling the y_n -separant of F correspond to solutions annulling the z -separant of \bar{F} .

The preceding proof shows that the theorem stated at the beginning of this note holds for \bar{F} . The correspondence just described shows that its truth for \bar{F} implies its truth for F . Hence it is true in general.

REFERENCES

1. J. F. Ritt, *Differential algebra*, Amer. Math. Soc. Colloquium Publications, vol. 33, 1950.
2. R. M. Cohn, *Manifolds of difference polynomials*, Trans. Amer. Math. Soc. vol. 64 (1948) pp. 133-172.
3. ———, *A note on the singular manifolds of a difference polynomial*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 917-922.
4. ———, *Extensions of difference fields*, Amer. J. Math. vol. 74 (1952) pp. 507-530.

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