COHOMOLOGY IN ABSTRACT UNIT GROUPS

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A. H. Clifford and S. MacLane [2] considered in 1941 the group of factor-sets $H^2(\Gamma, U)$ of a finite group Γ over its abstract unit group U. They proved the main theorem to the effect that $H^2(\Gamma, U)$ is isomorphic to the multiplicator M of Γ defined by I. Schur and also several other theorems under the assumption that Γ is a solvable group. They conjectured that these should hold for general finite groups Γ. In 1942 A. Weil proved the main theorem for general finite groups Γ, but this result was not published.[1] In this short note we shall prove that all the theorems in [2] are valid for general finite groups Γ, and also we shall extend their results for all (positive, zero, and negative) dimensional cohomology groups.[2]

1. We shall first prove a general lemma on cohomology groups. Let Δ be a finite group, and let E be a Δ-module. Suppose that A_1, A_2 are two Δ-submodules which are disjoint: $A_1 \cap A_2 = 0$. Then we have the following commutative diagram such that each row and each column are exact:

$$
(1) \qquad
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] \arrow[d] & A_1 \arrow[r, "j_{21}"] \arrow[d, "i_{12}"] & (A_1 + A_2)/A_2 \arrow[r] \arrow[d, "i_{13}"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "i_{22}"] \arrow[d, "j_{11}"] & E \arrow[r, "j_{22}"] \arrow[d, "j_{12}"] & E/A_2 \arrow[r] \arrow[d, "j_{13}"] & 0 \\
0 \arrow[r] & (A_1 + A_2)/A_1 \arrow[r, "i_{23}"] \arrow[d] & E/A_1 \arrow[r, "j_{23}"] \arrow[d] & E/(A_1 + A_2) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

Let us denote, in general, by $H^r(\Delta, A)$ the r-cohomology group of a group Δ over a Δ-module A.

LEMMA. *Assume that* $H^r(\Delta, E) = 0$ *for* $r = 0, \pm 1, \pm 2, \cdots$. *Then we have for all* $r = 0, \pm 1, \pm 2, \cdots$,

Received by the editors April 1, 1954.

[1] The author wishes to express his thanks to Professor Saunders MacLane who kindly told him this fact and allowed him to read an unpublished manuscript about it.

[2] For the definition of negative dimensional cohomology groups of a finite group and for the properties of cohomology groups see, for example, Artin-Tate [1].

$$\text{(I)}^* \quad H^r(\Delta, E/A_1) \overset{\delta}{\cong} H^{r+1}(\Delta, A_1) \overset{j^*}{\cong} H^{r+1}(\Delta, (A_1 + A_2)/A_2),$$

$$\text{(II)}^* \quad 0 \rightarrow H^r(\Delta, E/A_2) \xrightarrow{j^*} H^r(\Delta, E/(A_1 + A_2)) \xrightarrow{\delta} H^{r+1}(\Delta, (A_1 + A_2)/A_2) \xrightarrow{i^*} 0 \quad (\textit{exact})$$

and similar formulas hold by interchanging the subscripts 1 *and* 2,

$$\text{(III)}^* \quad H^r(\Delta, E/(A_1 + A_2)) = j_{23}^*(H^r(\Delta, E/A_1)) + j_{13}^*(H^r(\Delta, E/A_2)).$$

Proof. (i) (I)* is evident by our assumption $H^r(\Delta, E)=0$. (ii) Since $(i_{13})^*=(j_{22})^* \circ (i_{12})^* \circ (j_{21}^{-1})^*$ and $(i_{12})^*=0$ by our assumption, we have $(i_{13})^*=0$. Hence we get (II)* by the exact sequence of cohomology groups derived from the 3rd column of the diagram (1). (iii) From (1) follows

$$\text{(2)} \qquad \begin{array}{ccccccccc} & & 0 & \longrightarrow & H^r(\Delta, E/A_2) & \xrightarrow{\delta_1} & H^{r+1}(\Delta, A_2) & \xrightarrow{i_{22}^*} & 0 \\ & & & & \downarrow j_{13}^* & & \downarrow j_{11}^* & & \\ 0 \rightarrow H^r(\Delta, E/A_1) & \xrightarrow{j_{23}^*} & & & H^r(\Delta, E/(A_1+A_2)) & \xrightarrow{\delta_2} & H^{r+1}(\Delta, (A_1+A_2)/A_1) & \xrightarrow{i_{23}^*} & 0. \end{array}$$

Here j_{13}^* is an into-isomorphism and δ_1, j_{11}^* are onto-isomorphisms. Hence $j_{13}^*(H^r(\Delta, E/A_2))$ is a splitting system of representatives of $H^r(\Delta, E/(A_1+A_2))$ mod $j_{23}^*(H^r(\Delta, E/A_1))$. This proves (III)*, q.e.d.

2. Let Γ be a finite group of order n, and $\Gamma(Z)$ be its group ring over the integers Z. Put $u=\sum_{\sigma\in\Gamma}\sigma\in\Gamma(Z)$. Then by definition the factor group $U=\Gamma(Z)/Zu$ is the *abstract unit group* of Γ. Now let Δ be an arbitrary subgroup of Γ. Let us take $E=\Gamma(Z)$, $A_1 = \sum_{\sigma\neq 1} Z(1-\sigma)$, and $A_2=Zu$. Clearly $A_1 \cap A_2=0$. Since $E=\Gamma(Z)$ is Δ-free, the assumption in the lemma is satisfied. Hence we can apply the lemma. Here we may identify $E/A_1=Z$ and $j_{12}=\text{tr}$, where $\text{tr}\,(\sum_\sigma a_\sigma\cdot\sigma)=\sum_\sigma a_\sigma\in Z$ $(a_\sigma\in Z)$. Then the 3rd row of the diagram (1) may be replaced by

$$\text{(3)} \qquad 0 \rightarrow nZ \xrightarrow{i_{23}} Z \xrightarrow{j_{23}} Z/nZ \rightarrow 0$$

where Δ operates on these modules trivially. Also j_{13} and j_{11} become the homomorphism tr induced in U $(\rightarrow Z/nZ)$ and Zu $(\rightarrow nZ)$ respectively. Finally put $U_0=(A_1+A_2)/A_2$, which is the kernel of the mapping $\text{tr}\,U \rightarrow Z/nZ$. By these substitutions we have the following formulas from our lemma:

For all $r=0, \pm, \pm 2, \cdots$

$$(\mathrm{I})_1 \quad H^r(\Delta, U) \overset{\delta}{\cong} H^{r+1}(\Delta, Z),$$

$$(\mathrm{I})_2 \quad H^{r-1}(\Delta, Z) \overset{j_{21}^* \cdot \delta}{\cong} H^r(\Delta, U_0),$$

$$(\mathrm{II})_1 \quad 0 \rightarrow H^r(\Delta, U) \xrightarrow{\mathrm{tr}^*} H^r(\Delta, Z/nZ) \xrightarrow{\delta} H^{r+1}(\Delta, U_0) \xrightarrow{i^*} 0 \ (exact),$$

$$(\mathrm{II})_2 \quad 0 \rightarrow H^r(\Delta, Z) \xrightarrow{i^*} H^r(\Delta, Z/nZ) \xrightarrow{\delta} H^{r+1}(\Delta, nZ) \xrightarrow{i^*} 0 \ (exact),$$

$$(\mathrm{III}) \quad H^r(\Delta, Z/nZ) = j_{23}^*(H^r(\Delta, Z)) + \mathrm{tr}^*\,(H^r(\Delta, U)),$$

where Δ operates trivially on Z, nZ and Z/nZ.

Now we get several theorems in [2] as corollaries of these formulas. Namely, from $(\mathrm{I})_2$ follows

(i) $H^0(\Delta, U_0) \cong H^{-1}(\Delta, Z) = 0$; $H^1(\Delta, U_0) \cong H^0(\Delta, Z) \cong Z/mZ$ *where m is the order of Δ*; $H^2(\Delta, U_0) \cong H^1(\Delta, Z) = 0$ (formulas (1), (2) in §6 and corollary in §1 of [2]). From $(\mathrm{II})_1$ follows

(ii) tr^*: $H^2(\Delta, U) \rightarrow H^2(\Delta, Z/nZ)$ *is an into-isomorphism* (Theorem 1.A of [2]),

(iii) $i^*(H^1(\Delta, U_0)) = 0$ *in* $H^1(\Delta, U)$ (Theorem 1.B of [2]).

From $(\mathrm{II})_1$ and $H^2(\Delta, U_0) = 0$ follows

(iv) tr^*: $H^1(\Delta, U) \rightarrow H^1(\Delta, Z/nZ)$ *is an onto-isomorphism* (Theorem 2.B of [2]).

3. Let Ω be an algebraically closed field of characteristic not dividing the order n of Γ. Then the *multiplicator* M of Γ is defined by I. Schur as $M = H^2(\Gamma, \Omega^*)$, where Γ acts trivially on the multiplicative group Ω^*. Let W be the group of all the roots of unity in Ω. Consider the exact sequence $1 \rightarrow W \rightarrow \Omega^* \rightarrow \Omega^*/W \rightarrow 1$. Since the group Ω^*/W is uniquely divisible, so $H^r(\Delta, \Omega^*/W) = 0$ for all r. Hence we have $M = H^2(\Gamma, \Omega^*) \cong H^2(\Gamma, W) \cong H^2(\Delta, Q/Z)$, where Q is the additive group of rationals. Let the homomorphism aver. (=average) be defined on $\Gamma(Z)$ by

$$\mathrm{aver.}\left(\sum_\sigma a_\sigma \cdot \sigma\right) = \frac{1}{n}\sum_\sigma a_\sigma = \frac{1}{n}\,\mathrm{tr}\left(\sum_\sigma a_\sigma \cdot \sigma\right) \in Q.$$

This homomorphism aver. induces also the homomorphism aver.: $U \rightarrow Q/Z$. A. Weil proved the main theorem in [2] for general Γ in the form:

(v) aver.*: $H^2(\Gamma, U) \rightarrow H^2(\Gamma, Q/Z)$ *is an onto-isomorphism.* For the sake of completeness we shall give here a proof which is essentially the same as that of A. Weil. Let us consider the commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & Zu \arrow[r] \arrow[d, "\phi_1"] & \Gamma(Z) \arrow[r] \arrow[d, "\phi_2"] & U \arrow[r] \arrow[d, "\phi_3"] & 0 \\
0 \arrow[r] & Z \arrow[r] & Q \arrow[r] & Q/Z \arrow[r] & 0
\end{tikzcd}
\tag{4}
$$

where ϕ = aver. Since $H^r(\Delta, Q) = H^r(\Delta, \Gamma(Z)) = 0$ for all r, this diagram induces the commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & H^r(\Delta, U) \arrow[r, "\delta_1"] \arrow[d, "\phi_3^*"] & H^{r+1}(\Delta, Zu) \arrow[r] \arrow[d, "\phi_1^*"] & 0 \\
0 \arrow[r] & H^r(\Delta, Q/Z) \arrow[r, "\delta_2"] & H^{r+1}(\Delta, Z) \arrow[r] & 0.
\end{tikzcd}
\tag{5}
$$

Here ϕ_1^* is an onto-isomorphism, so is $\phi_3^* = \delta_2^{-1} \circ \phi_1^* \circ \delta_1$. Hence we get

(IV) aver.*: *$H^r(\Delta, U) \rightarrow H^r(\Delta, Q/Z)$ is an onto-isomorphism for all r.*

The relation between tr* and aver.* on $H^r(\Delta, U)$ is given as follows. Let ψ be the homomorphism defined by $\psi(\alpha) = \alpha \times (1/n)$ on $Z(\rightarrow Q)$, $nZ(\rightarrow Z)$ and $Z/nZ(\rightarrow Q/Z)$ respectively. Then we have aver. $= \psi \circ$ tr, and ψ induces the homomorphism: $H^r(\Delta, Z/nZ) \xrightarrow{\psi^*} H^r(\Delta, Q/Z)$. Then we have

(V) *$\psi^* \circ j_{23}^*(H^r(\Delta, Z)) = 0$ and ψ^* is an isomorphism of* tr* $(H^r(\Delta, U))$ *onto $H^r(\Delta, Q/Z)$.*

Proof. Let us consider the commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & nZ \arrow[r, "i_1"] \arrow[d, "\psi_1"] & Z \arrow[r, "j_1"] \arrow[d, "\psi_2"] & Z/nZ \arrow[r] \arrow[d, "\psi_3"] & 0 \\
0 \arrow[r] & Z \arrow[r, "i_2"] & Q \arrow[r, "j_2"] & Q/Z \arrow[r] & 0.
\end{tikzcd}
$$

Since $(\psi_2)^* = 0$ by $H^r(\Delta, Q) = 0$, we have $\psi_3^* \circ j_1^* = j_2^* \circ \psi_2^* = 0$. Since $\phi_3^* = \psi_3^* \circ j_{13}^*$ and ϕ_3^* (j_{13}^*) is an isomorphism-onto (-into), so ψ_3^* is an onto-isomorphism, q.e.d. These considerations actually cover Theorem 2.A of [2].

References

1. E. Artin and J. Tate, *Algebraic functions and algebraic numbers*, Lecture notes at Princeton University, vol. 2, to be published.

2. A. H. Clifford and S. MacLane, *Factor-sets of a group in its abstract unit group*, Trans. Amer. Math. Soc. vol. 50 (1941) pp. 385–406.

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