BANACH SPACES AND ALGEBRAS OF CONTINUOUS FUNCTIONS¹

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1. Introduction. If X is a Hausdorff space, then we shall denote by $C(X)$ the set of real, bounded, continuous functions defined on X . If we define addition and scalar multiplication of functions in the usual way, then $C(X)$ becomes a group with real operator domain. The function which is identically zero on X is the additive identity of this group, and we denote it by θ . Under the norm $\|f\| = \sup_X |f(x)|$ this space becomes a real Banach space. Finally, provided with the usual multiplication of functions, this space becomes a real Banach algebra. The function which is identically one on X is the multiplicative identity, and we denote it by e .

According to the Banach-Stone theorem [5],² if X is compact, then the topology of X is reflected in the algebraic and metric structure of $C(X)$ in the sense that if $C(X)$ and $C(Y)$ are equivalent as Banach spaces, then X is homeomorphic to Y . This general structural relation suggests particularization: what are the specific algebraic or metric properties of $C(X)$ which correspond to specific topological features of X ? Significant results have been obtained in this direction by Myers, Eilenberg, and others [1; 2; 3; 4].

This paper is drawn along these lines, and is concerned in particular with those spaces of continuous functions which are defined on the topological product of two compact Hausdorff spaces, or on compact Hausdorff spaces which contain such a product.³ The central problems are those of characterization; for the cases of products, and spaces which contain a product subspace, characterizations have been obtained. I have also obtained characterizations of continuous functions spaces over fiber spaces and fiber bundles, but the results are quite technical and not very revealing, and therefore are not of sufficient interest to include here.

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² Numbers in brackets refer to the bibliography.

³ In this paper the only continuous function spaces we shall consider are those defined over compact Hausdorff spaces.

2. Subspaces of $C(X)$.

DEFINITION. A set of elements T drawn from a Banach space B maximal with respect to the property $b_1, \dots, b_n \in T \rightarrow \|\sum_{i=1}^n b_i\| = \sum_{i=1}^n \|b_i\|$ is called a T -set of B .

The T -sets of a Banach space were introduced by Myers [2], and their significant features were enunciated by him. We shall use that T -sets exist, cover the space B , are positively homogeneous, and are convex. Also, if the Banach space B is a space of continuous functions, say $B = C(X)$, where X is compact, then Myers has shown that every T -set in B has the form

$$\{f \in B \mid f(x_0) = \|f\|\} \quad \text{or} \quad \{f \in B \mid f(x_0) = -\|f\|\},$$

and conversely, every such subset is a T -set of B . If we denote the first of these T -sets by $T(x_0)$, then it is easily seen that the second has the form $-T(x_0)$.

DEFINITION. A subspace F of a Banach space B is said to be T -normally imbedded in B if for every T -set T of B , $T \cap F$ is a T -set of F .

We shall be concerned with the T -normal subspaces of a Banach space, and criteria for recognizing them are therefore of interest:

T1. A subspace F is T -normally imbedded in B if and only if every T -set of B contains a T -set of F .

T2. If F is a subspace of $C(X)$ which contains the constant functions, and is closed under absolute value, then F is T -normal in $C(X)$.

The first of these conditions follows almost immediately from the maximal character of T -sets, and to prove the second we proceed as follows:

Suppose that F is not T -normal in $C(X)$. Then there exists a T -set T of $C(X)$ such that $T \cap F$ is not a T -set of F . Thus, there exists $f \in F$ such that $\|f+g\| = \|f\| + \|g\|$ for every $g \in T \cap F$, but $f \notin T$. Let us suppose that $T = T(x_0)$ (the argument is similar if $T = -T(x_0)$).

Now, $|f(x_0)| < \|f\|$, for if $f(x_0) = \|f\|$ then $f \in T$, contrary to assumption, and if $-f(x_0) = \|f\|$, then $-f \in T$ so that $\|f+(-f)\| = 2\|f\| = 0$, whence $f = \theta$, a contradiction since θ is contained in every T -set. We can suppose that $f(x_0) \geq 0$, for the argument is strictly analogous if $f(x_0) \leq 0$.

Let $h(x) = f(x) - f(x_0)$, $m(x) = |h(x)|$, and $n(x) = \|m\| - m(x)$. Since F is a subspace which contains the constant functions and is closed under absolute value, the functions h , m , and n are contained in F , and $n \neq \theta$.

Now one easily checks that $n(x_0) = \|n\|$ so that $n \in T \cap F$, whence by assumption $\|f+n\| = \|f\| + \|n\|$. However this relation implies that both f and n are in some T -set, and since $n(x) \geq 0$, it must have the form $T(\bar{x})$. Hence $f(\bar{x}) = \|f\|$, and $n(\bar{x}) = \|n\|$. But we then have $f(\bar{x}) > f(x_0)$ so that both $h(\bar{x}) > 0$ and $m(\bar{x}) > 0$. This however yields $n(\bar{x}) < \|n\|$, a contradiction, and the theorem is therefore proved.

DEFINITION. If T is a T -set of a Banach space B , then the set $S = T \cup -T$ is called a *symmetricized T -set* of B , denoted by an sT -set.

One easily verifies that a T -normal subspace is sT -normal. More precisely:

T3. If F is a T -normal subspace of B , and S is an sT -set of B , then $S \cap F$ is an sT -set of F .

Our concern now will not be so much with the orientation of a single subspace in a Banach space, but rather with the mutual orientation of several subspaces with each other. In particular three possible mutual orientations will be interesting:

DEFINITION. If F and G are closed subspaces of a Banach space B with the property that $f \in F, g \in G$ imply either $\|f+g\| = \|f\| + \|g\|$ or $\|f-g\| = \|f\| + \|g\|$, then we say that F and G are *additively related*.

DEFINITION. If F and G are closed subspaces of a Banach space B with the property that for any pair of sT -sets S_F, S_G of F and G respectively there exists an sT -set S of B such that $S_F \cup S_G \subset S$, then we say that F and G are *S -admittant in B* .

DEFINITION. If F and G are closed subspaces of a Banach space B with the property that for any pair of sT -sets S_F, S_G of F and G respectively there exists not more than one sT -set S of B such that $S_F \cup S_G \subset S$, then we say that F and G are *S -determinant in B* .

These definitions are not unrelated, in fact

T4. If F and G are additively related subspaces of $C(X)$, then they are *S -admittant in $C(X)$* .

PROOF. Let us suppose that F and G are additively related but are not S -admittant. Then there exist sT -sets S_F and S_G of F and G respectively such that $S_F \cup S_G$ is contained in no sT -set of $C(X)$.

Let $S_F = T_F \cup -T_F$ and $S_G = T_G \cup -T_G$. Note that neither $T_F \cup T_G$ nor $T_F \cup -T_G$ is contained in a T -set of $C(X)$ for otherwise we should have $S_F \cup S_G$ contained in an sT -set of $C(X)$. Hence, there exist $f_F, \bar{f}_F \in T_F$ and $f_G, \bar{f}_G \in T_G$ such that $\|f_F + f_G\| < \|f_F\| + \|f_G\|$ and $\|\bar{f}_F - \bar{f}_G\| < \|\bar{f}_F\| + \|\bar{f}_G\|$. If we let $g_F = f_F + \bar{f}_F$ and $h_G = f_G + \bar{f}_G$, a simple computation shows that $\|g_F + h_G\| < \|g_F\| + \|h_G\|$ and $\|g_F - h_G\| < \|g_F\| + \|h_G\|$. Since however $g_F \in F$ and $h_G \in G$ we see that F and G

are not additively related, contrary to assumption. Hence the theorem is proved.

Now we turn to an investigation of these concepts in terms of the algebraic structure of $C(X)$.

3. Subalgebras of $C(X)$.

DEFINITION. A subalgebra of $C(X)$ which contains the unit e is called an *analytic subalgebra* of $C(X)$.

T5. If F is an analytic subalgebra of $C(X)$, then F is T -normal in $C(X)$.

PROOF. We assume that F is not T -normal, and then as in T2 we establish the existence of a function f such that $\|f+g\| = \|f\| + \|g\|$ for every $g \in T \cap F$, but $f \notin T$. If we let $T = T(x_0)$, then we may assert that $|f(x_0)| < \|f\|$.

We now let $h(x) = f(x) - f(x_0)$, $m(x) = (h(x))^2$, and $n(x) = \|m\| - m(x)$. Proceeding exactly as in T2 we show that there exists $\bar{x} \in X$ such that $f(\bar{x}) = \|f\|$ and $n(\bar{x}) = \|n\|$. By the construction of n however we have $n(\bar{x}) < \|n\|$, and this contradiction establishes the theorem.

We now turn to characterizing S -determinacy within the algebraic and metric structure of $C(X)$.

DEFINITION. A subset N of the space $C(X)$ is said to *separate* X if for every x and $\bar{x}$ in X there exists a function f in N such that $f(x) \neq f(\bar{x})$.

T6. If A_1 and A_2 are analytic subalgebras of $C(X)$, then A_1 and A_2 are S -determinant in $C(X)$ if and only if $A_1 \cup A_2$ separates X .

PROOF. (a) Suppose that $A_1 \cup A_2$ separates X . Let S_1 and S_2 be sT -sets of A_1 and A_2 respectively, and suppose that $S_1 \cup S_2 \subset S(\bar{x})$, and $S_1 \cup S_2 \subset S(x^*)$, where $x \neq x^*$. Since $A_1 \cup A_2$ separates X , there exists $g \in A_1 \cup A_2$, say $g \in A_1$, such that $g(\bar{x}) \neq g(x^*)$. Let $h(x) = -[g(x) - g(\bar{x})]^2$, and let $f(x) = h(x) + \|h\|$. Since A_1 is analytic, f is contained in A_1 , and by virtue of the definition of f , $f(\bar{x}) = \|f\|$, $f(x^*) < \|f\|$, and $f(x) \geq 0$. Therefore $f \in S(\bar{x}) \cap A_1 = S_1$, and $f \notin S(x^*)$. Hence $S_1 \cup S_2 \not\subset S(x^*)$, a contradiction. We have then that $S_1 \cup S_2$ is contained in not more than one sT -set of $C(X)$.

(b) Suppose that for any pair of sT -sets S_1 and S_2 of A_1 and A_2 respectively there exists not more than one sT -set S of $C(X)$ such that $S_1 \cup S_2 \subset S$. Let $\bar{x}, x^* \in X$, $\bar{x} \neq x^*$, and let $\bar{S}_i = S(\bar{x}) \cap A_i$ and $S_i^* = S(x^*) \cap A_i$, $i = 1, 2$. Now, by assumption either $\bar{S}_1 \neq S_1^*$ or $\bar{S}_2 \neq S_2^*$, say $\bar{S}_1 \neq S_1^*$. Hence, there exists $f \in \bar{S}_1$, $f \notin S_1^*$ or there exists $f \in S_1^*$, $f \notin \bar{S}_1$. Suppose the first case obtains. But $f \in \bar{S}_1$ implies that

$|f(\bar{x})| = \|f\|$, and $f \in S_1^*$ implies that $|f(x^*)| < \|f\|$, so that f separates $\bar{x}$ and x^* .

DEFINITION. If N is a subset of the algebra $C(X)$, let $(N)_A$ denote the smallest subalgebra of $C(X)$ which contains N .

T7. *A subset F of $C(X)$ which contains e separates X if and only if $(F)_A$ is dense in $C(X)$.*

PROOF. (a) If F separates X , by the Weierstrass-Stone theorem [5], $(F)_A$ is dense in $C(X)$.

(b) If F does not separate X , there exist $x, \bar{x}$ contained in X , $x \neq \bar{x}$, such that $f(x) = f(\bar{x})$ for every f contained in F . Hence $q(x) = q(\bar{x})$ for every $q \in (F)_A$. Since however $C(X)$ is completely regular it follows that $(F)_A$ is not dense in $C(X)$.

Now we may summarize the results of the last several theorems in a rather nice characterization of S -determinacy within $C(X)$:

T8. *If A_1 and A_2 are analytic subalgebras of $C(X)$, then A_1 and A_2 are S -determinant in $C(X)$ if and only if the closure of the algebra generated by A_1 and A_2 is $C(X)$.*

The proof of this theorem follows from the results of T6 and T7.

4. **The algebra $C(X)$ where X is a topological product.** In this section we shall let X_1 and X_2 be compact Hausdorff spaces, and shall let X denote their topological product. In accord with the previous notation the spaces (algebras) of real continuous functions defined on these spaces will be denoted by $C(X)$ and $C(X_i)$, $i=1, 2$.

DEFINITION. $g \in C(X)$ will be called a *semi-constant function of type (1)* if

$$(x_1, x_2), (x_1, \bar{x}_2) \in X \rightarrow g(x_1, x_2) = g(x_1, \bar{x}_2),$$

and of *type (2)* if

$$(x_1, x_2), (\bar{x}_1, x_2) \in X \rightarrow g(x_1, x_2) = g(\bar{x}_1, x_2).$$

Let F_i denote the collection of semi-constant functions of type (i).

One may easily verify that the collections F_i are analytic subalgebras of $C(X)$, and then from the observation that each F_i is equivalent as a Banach algebra to $C(X_i)$ we may conclude that the F_i are complete, and therefore closed in $C(X)$. For purposes of reference then:

T9. *The collections F_i of semi-constant functions of type (i) are closed subalgebras of $C(X)$, $i=1, 2$.*

DEFINITION. Let $f \in C(X)$. Then the *norm set* of f , $N(f) = \{x \in X \mid |f(x)| = \|f\|\}$.

T10. F_1 and F_2 are additively related.

PROOF. Let $f_i \in F_i$, $i=1, 2$. Since X is compact, there exist $x, y \in X$ such that $|f_1(x)| = \|f_1\|$ and $|f_2(y)| = \|f_2\|$. If now $x = (x_1, x_2)$ and $y = (y_1, y_2)$ we have from the definition of the F_i that $|f_1(x_1, y_2)| = |f_1(x_1, x_2)| = \|f_1\|$, and $|f_2(y_1, y_2)| = |f_2(x_1, y_2)| = \|f_2\|$, or that $\bar{x} = (x_1, y_2) \in N(f_1) \cap N(f_2)$. Now there are essentially two cases: where both $f_1(\bar{x})$ and $f_2(\bar{x})$ are non-negative, and where one of $f_1(\bar{x})$, $f_2(\bar{x})$ is negative.

(a) If $f_1(\bar{x}) \geq 0$ and $f_2(\bar{x}) \geq 0$, then $f_1(\bar{x}) = \|f_1\|$ and $f_2(\bar{x}) = \|f_2\|$ so that $f_1, f_2 \in T(\bar{x})$, whence $\|f_1 + f_2\| = \|f_1\| + \|f_2\|$.

(b) If $f_1(\bar{x}) \geq 0$ and $f_2(\bar{x}) \leq 0$, then $f_1(\bar{x}) = \|f_1\|$ and $-f_2(\bar{x}) = \|f_2\|$ so that $f_1, -f_2 \in T(\bar{x})$, whence $\|f_1 - f_2\| = \|f_1 + (-f_2)\| = \|f_1\| + \|f_2\| = \|f_1\| + \|f_2\|$.

Thus, for every pair of functions f_1 and f_2 contained respectively in F_1 and F_2 we have either $\|f_1 + f_2\| = \|f_1\| + \|f_2\|$ or $\|f_1 - f_2\| = \|f_1\| + \|f_2\|$, so that the algebras F_1 and F_2 are additively related.

T11. F_1 and F_2 are S -determinant in $C(X)$.

PROOF. By virtue of T6 and T9 we need only show that $F_1 \cup F_2$ separates X . Let $x, \bar{x} \in X$, $x \neq \bar{x}$, where $x = (x_1, x_2)$, $\bar{x} = (\bar{x}_1, \bar{x}_2)$. Since either $x_1 \neq \bar{x}_1$, or $x_2 \neq \bar{x}_2$, let us suppose that $x_1 \neq \bar{x}_1$. Because $C(X_1)$ is completely regular, there exists $\bar{f} \in C(X_1)$ such that $\bar{f}(x_1) \neq \bar{f}(\bar{x}_1)$. We define $f(x_1, x_2) = \bar{f}(x_1)$, and note that $f \in F_1$. Since, however, $f(x_1, x_2) = \bar{f}(x_1) \neq \bar{f}(\bar{x}_1) = f(\bar{x}_1, \bar{x}_2)$, f separates x and $\bar{x}$ and the theorem is therefore proved.

Thus, to summarize briefly: the collections F_i of semi-constant functions of type (i) on X are additively related, S -determinant closed analytic subalgebras of $C(X)$.

5. Characterization of the space (algebra) $C(X_1 \times X_2)$. In this section we shall let $\mathcal{S}$ denote the class of sT -sets of $C(X)$, and $\mathcal{S}_i$ the class of sT -sets of a closed subspace F_i of $C(X)$. For $f \in C(X)$ and $f_i \in F_i$ let $C(f) = \{S \in \mathcal{S} \mid f \in S\}$, and $C_i(f_i) = \{S_i \in \mathcal{S}_i \mid f_i \in S_i\}$. Let $\Lambda = \{C(f) \mid f \in C(X)\}$, and $\Lambda_i = \{C_i(f_i) \mid f_i \in F_i\}$.

The collections of subsets Λ and Λ_i of $\mathcal{S}$ and $\mathcal{S}_i$ respectively may be considered as generators for systems of closed subsets of $\mathcal{S}$ and $\mathcal{S}_i$ respectively, thus providing these sets with topologies. We shall call these the normal topologies, and shall assume hereafter that $\mathcal{S}$ and $\mathcal{S}_i$ carry the normal topologies.

T12. The space $\mathcal{S}$ is homeomorphic to X .

PROOF. Except for changes in notation this result is contained in Eilenberg's paper [1].

T13. Let $C(X)$ be the Banach space of real continuous functions defined on the compact space X . Then X is a product space if and only if $C(X)$ contains two T -normal, additively related closed subspaces which are S -determinant in $C(X)$.

PROOF. The necessity of these conditions has been established in T9, T5, T10, and T11.

By T12 the sufficiency of the conditions will be demonstrated if we can show that $\mathcal{S}$ is a product space. We shall do this by exhibiting a homeomorphism of $\mathcal{S}$ onto $\mathcal{S}_1 \times \mathcal{S}_2$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are the classes of sT -sets of the subspaces F_1 and F_2 whose existence is asserted in the hypotheses of the theorem.

We define the mapping $F: \mathcal{S} \rightarrow \mathcal{S}_1 \times \mathcal{S}_2$, where $F(S) = (S \cap F_1, S \cap F_2)$. By the T -normality of the F_i , this mapping is well defined (see T3).

(a) F is onto all of $\mathcal{S}_1 \times \mathcal{S}_2$: By T4, F_1 and F_2 are S -admittant in $C(X)$. Hence $S_i \in \mathcal{S}_i$, $i=1$ and 2 , implies there exists $S \in \mathcal{S}$ such that $S_1 \cup S_2 \subset S$. Now, I claim that $S_1 = S \cap F_1$ and $S_2 = S \cap F_2$, for suppose that $S_1 = T_1 \cup -T_1$ and $S = T \cup -T$. Then $T_1 \cup -T_1 \subset T \cup -T$ from which we may conclude that either $T_1 \subset T$ or $T_1 \subset -T$ for, if not, there exists $f \in T_1$, $f \notin T$, and there exists $g \in T_1$, $g \notin -T$. Since T_1 is positively homogeneous $f+g \in T_1$, and hence $f+g$ must be contained in either T or $-T$. Suppose that $f+g \in T$. Since $-f \in T$ we have $\|(f+g) + (-f)\| = \|f+g\| + \|-f\| = 2\|f\| + \|g\| = \|g\|$, so that $f = \theta$, a contradiction since $\theta \in T$. On the other hand suppose that $f+g \in -T$. Since $-g \in -T$ we have similarly that $\|(f+g) + (-g)\| = \|f+g\| + \|-g\| = \|f\| + 2\|g\| = \|f\|$, so that $g = \theta$, a contradiction since $\theta \in -T$. We have then that either $T_1 \subset T$ or $T_1 \subset -T$. Let us suppose that $T_1 \subset T$ whence $T_1 \subset T \cap F_1$. Now, by the additivity of $T \cap F_1$, and the maximality of T_1 , $T_1 = T \cap F_1$. Hence, $T_1 \cup -T_1 = (T \cap F_1) \cup (-T \cap F_1) = (T \cup -T) \cap F_1 = S \cap F_1$, so that $S_1 = S \cap F_1$. Analogously we have that $S_2 = S \cap F_2$. Hence, $F(S) = (S_1, S_2)$, and F is therefore onto.

(b) F is 1-1: If $S \cap F_i = \bar{S} \cap F_i$ for $i=1$ and 2 , then $(S \cap F_1) \cup (S \cap F_2) \subset S$ and $(\bar{S} \cap F_1) \cup (\bar{S} \cap F_2) \subset \bar{S}$. But, by the T -normality of the F_i , $S \cap F_i \in \mathcal{S}_i$ for $i=1$ and 2 , and hence by the S -determinancy of the F_i , $S = \bar{S}$. Hence $F(S) = F(\bar{S})$ implies that $S = \bar{S}$, and F is therefore 1-1.

Now, since $\mathcal{S}$ is compact, in order to show that F is a homeomor-

phism it is sufficient to show that it is continuous.

(c) F is continuous: We shall show that the inverse image of any set in the closed sub-base of $\mathcal{S}_1 \times \mathcal{S}_2$ is a member of the closed base of $\mathcal{S}$. In particular, for $f_i \in F_i$, $i=1$ and 2 , we shall show that there exists $f \in C(X)$ such that $C(f) = F^{-1}(C_1(f_1), C_2(f_2))$.

Since the F_i are additively related, either

$$\|f_1 + f_2\| = \|f_1\| + \|f_2\|, \quad \text{or} \quad \|f_1 - f_2\| = \|f_1\| + \|f_2\|.$$

(1) Suppose that $\|f_1 + f_2\| = \|f_1\| + \|f_2\|$. In this case we let $f = f_1 + f_2$, and must show then that $C(f) = F^{-1}(C_1(f_1), C_2(f_2))$.

(i) $F^{-1}(C_1(f_1), C_2(f_2)) \subset C(f)$: Let $S \in F^{-1}(C_1(f_1), C_2(f_2))$. Then $f_i \in S \cap F_i$, and hence $f_1, f_2 \in S$. Since $\|f_1 + f_2\| = \|f_1\| + \|f_2\|$, f_1 and f_2 are in the same T -set of S . Hence $f_1 + f_2$ is in this T -set, and therefore in S . But $f \in S$ means that $S \in C(f)$.

(ii) $C(f) \subset F^{-1}(C_1(f_1), C_2(f_2))$. Let $S \in C(f)$. Hence $f = f_1 + f_2 \in S$. Now, since $F(S) = (S \cap F_1, S \cap F_2)$, if we can show that f_1 and f_2 are in S we are through, for $f_i \in F_i$, $f_i \in S$ implies that $f_i \in S \cap F_i$ which implies that $S \cap F_i$ are contained in $C_i(f_i)$ which in turn implies that $S \in F^{-1}(C_1(f_1), C_2(f_2))$.

We shall show that $f_1 \in S$, and note that the argument is similar for f_2 .

Let $S = T \cup -T$, and suppose that $f \in T$. Then f_1 will be in T , and therefore in S , if $g \in T$ implies that $\|f_1 + g\| = \|f_1\| + \|g\|$. This we shall show. Let $g \in T$. Now, $\|g + f_1\| = \|g + (f_1 + f_2) - f_2\| \geq \|g + (f_1 + f_2)\| - \|f_2\| = \|g\| + \|f_1\|$, so that $\|g + f_1\| = \|g\| + \|f_1\|$.

(2) Suppose that $\|f_1 - f_2\| = \|f_1\| + \|f_2\|$. Rewriting this relation, $\|f_1 + (-f_2)\| = \|f_1\| + \|-f_2\|$, whence by (1) we can assert that $C(f_1 + (-f_2)) = F^{-1}(C_1(f_1), C_2(-f_2))$. But since $f_i \in S_i$ implies that $-f_i \in S_i$, $C_2(f_2) = C_2(-f_2)$, we have that $C(f_1 - f_2) = F^{-1}(C_1(f_1), C_2(f_2))$ which is the desired relation.

Thus, the proof is complete.

From T12 and the proof of T13 it is apparent that if the F_i are equivalent as Banach spaces to $C(X_i)$, then X is homeomorphic to $X_1 \times X_2$. Of course, every space is trivially a product space, that is the product of itself with a single point, and this case arises when one of the F_i is equivalent to the real numbers, and the other to $C(X)$. Thus, to eliminate this trivial case, we simply insist that neither of the F_i be all of $C(X)$.

The characterization of Banach spaces of the form $C(X_1 \times X_2)$ as expressed in T12, although satisfactory, involves somewhat artificial conditions: T -normality, and especially S -determinancy. When the

algebraic structure of the space is brought into play, however, a far nicer characterization is possible.

T14. *Let $C(X)$ be the Banach algebra of real, continuous functions defined on the compact space X . Then X is a product space if and only if $C(X)$ contains two analytic subalgebras which are additively related, and the closure of whose generated subalgebra is $C(X)$.*

PROOF. The necessity of these conditions has been demonstrated in T9, T10, T11, and T8.

The sufficiency will be established if we can show that these conditions imply the hypotheses of T12. This follows however from T5 and T8.

The next problem we shall consider is that of characterizing the algebra of real continuous functions defined on a compact space X which contains a closed subspace Y in the form of a topological product. This problem however necessitates a brief investigation of the ideal structure in the algebra $C(X)$; we turn to this now.

6. Ideals in $C(X)$. Because frequent reference will be made to some basic properties of function algebras developed by Stone [5], we shall summarize some of his results in the form of a theorem, T15.

T15. *Let $C(X)$ be an algebra of continuous functions on a compact space. Then corresponding to every closed ideal I in $C(X)$ there exists a closed subset $K \subset X$ such that $I = \{f \in C(X) \mid f(K) = 0\}$, and conversely for every closed subset $K \subset X$ the set $\{f \in C(X) \mid f(K) = 0\}$ is a closed ideal in $C(X)$. The maximal ideals then have the form $I = I(x_0) = \{f \in C(X) \mid f(x_0) = 0\}$, and every closed ideal is the intersection of all maximal ideals in which it is contained. Moreover, if $I = I(K)$ is a closed ideal in $C(X)$, and we provide the quotient algebra $C(X)/I$ with the norm $\|b\| = \inf_{c \in b} \|c\|$, where b is a coset in $C(X)/I$, then $C(X)/I$ is a Banach algebra equivalent to $C(K)$.*

Now, let $\mathfrak{I}$ denote the class of maximal ideals in $C(X)$, and let I be some closed ideal in $C(X)$. Let $\mathfrak{I}_1 = \{I_\alpha \in \mathfrak{I} \mid I \subset I_\alpha\}$ and let $\mathfrak{I}_2 = \{I_\alpha \in \mathfrak{I} \mid I_\alpha \not\subset \mathfrak{I}_1\}$. We define $I_1 = \bigcap I_\alpha \mid I_\alpha \in \mathfrak{I}_1$ and $I_2 = \bigcap I_\alpha \mid I_\alpha \in \mathfrak{I}_2$. By T14, we can assert that $I = I_1$. The closed ideal I_2 we shall call the ideal *complementary* to I , and denote it by I_c . Since $C(X)$ is without radical we might note that $C(X)$ and (θ) are mutually complementary ideals.

There are some elementary properties of these ideals which follow almost immediately from the above definitions.

T16. *Let $I = I(x)$ be a maximal ideal and $I' = (K)$ a closed ideal in*

$C(X)$, where K is closed in X . Let W be any subset of X . Then,

- (1) $I' \subset I$ if and only if $x \in K$,
- (2) $I(W) = I(\mathcal{C}(W))$, where $I(W) = \{f \in C(X) \mid f(W) = 0\}$, and $\mathcal{C}(W)$ denotes the closure of W ,
- (3) $I'_c = I(\mathcal{C}(CK))$, where CK is the complement in X of K .

PROOF. (1) follows immediately from the normality of the space X . Since $W \subset \mathcal{C}(W)$ implies that $I(\mathcal{C}(W)) \subset I(W)$, to obtain (2) we need only show that $I(W) \subset I(\mathcal{C}(W))$. This however follows from the continuity of the functions in $I(W)$. For (3) we may use the result (1): $I' \not\subset I(x) \Leftrightarrow x \notin K \Leftrightarrow x \in CK$, so that $I'_c = \bigcap I_\alpha \mid I_\alpha \in \mathfrak{I}_1 = \bigcap I_\alpha \mid I' \not\subset I_\alpha = \bigcap I(x) \mid x \in CK = I(CK)$, whence by (2) $I'_c = I(\mathcal{C}(CK))$.

Now the importance of these observations is expressed in the

COROLLARY. If $I = I(x)$ is a maximal ideal, and $I' = I(K)$ is a closed ideal in $C(X)$, where K is closed in X , then $x \in \text{int } K$ if and only if $I'_c \not\subset I$.

PROOF. By (3) of T16, $I'_c = I(\mathcal{C}(CK))$, and hence by (1) and the definition of the interior, $I'_c \not\subset I(x) \Leftrightarrow x \notin \mathcal{C}(CK) \Leftrightarrow x \in C(\mathcal{C}(CK)) \Leftrightarrow x \in \text{int } K$.

For purposes of reference we express this relation somewhat differently:

T17. Let $I' = I(K)$ be a closed ideal in $C(X)$ where K is closed in X . Then $\text{int } K \neq \emptyset$ if and only if $I'_c \neq (\theta)$.

PROOF. $\text{int } K \neq \emptyset \Leftrightarrow$ there exists $x \in \text{int } K \Leftrightarrow$ there exists $I(x)$ such that $I'_c \not\subset I(x)$ by the corollary of T16. But equivalently then $I'_c \neq (\theta)$, and the theorem is proved.

Note that T17 extends Stone's result as follows: the one-to-one correspondence between closed subsets of X and the closed ideals of $C(X)$ as expressed in T15 establishes a one-to-one correspondence between those closed subsets of X which have nonvacuous interiors and those ideals of $C(X)$ with non-null complements.

7. The algebra $C(X)$ where X contains a topological product. In this section we shall be concerned with the characterization of the algebra of real continuous functions defined on a compact space X which contains a closed subspace Y homeomorphic to a topological product space. This problem is of course almost nonsense unless we have a method for determining the relative "size" of the subspace Y in X . Thus, if Y consists of four distinct points it is homeomorphic to a product space, and therefore X contains a subset of the desired kind whenever it contains four or more points. We want to rule out this trivial case, and shall be insisting that Y contain an interior in X .

Let Y be a closed subspace of X , and let h be a homeomorphism of $X_1 \times X_2$ onto Y , where X_1 and X_2 are compact spaces. Let $I = I(Y)$ be the closed ideal in $C(X)$ of all functions which vanish on Y . Then $C(X)/I$ is equivalent to $C(Y)$ as a Banach algebra, and hence by T13 contains two analytic subalgebras F_1^* and F_2^* which are additively related and the closure of whose generated subalgebra is $C(X)/I$. Let $H: C(X) \rightarrow C(X)/I$ be the natural homomorphism and let F_1 and F_2 be the analytic subalgebras $H^{-1}(F_1^*)$ and $H^{-1}(F_2^*)$ respectively.

T18. $\mathcal{C}((F_1 \cup F_2)_A) = C(X)$.

PROOF. Let $f \in C(X)$, and $\epsilon > 0$. We shall show that there exists $g \in (F_1 \cup F_2)_A$ such that $\|f - g\| < \epsilon$. Since $\mathcal{C}((F_1^* \cup F_2^*)_A) = C(X)/I$, there exists $g^* \in (F_1^* \cup F_2^*)_A$ such that $\|g^* - H(f)\| < \epsilon$, i.e. $\sup_Y |g^*(y) - f(y)| < \epsilon$. Let h^* be the function defined on Y as follows: $h^*(y) = g^*(y) - f(y)$, and note that $\sup_Y |h^*(y)| < \epsilon$. Since h^* is defined on a closed subset of the normal space X , there exists a norm preserving extension h of h^* over X . If we let $g(x) = h(x) + f(x)$, then $g(y) = g^*(y)$ so that g extends g^* , and

$$\|f - g\| = \sup_X |f(x) - g(x)| = \sup_X |h(x)| < \epsilon.$$

Since however $H(g) = g^* \in (F_1^* \cup F_2^*)_A$, $g \in H^{-1}((F_1^* \cup F_2^*)_A) = (F_1 \cup F_2)_A$, and the theorem is therefore proved.

Now it turns out that the existence of these algebras with the properties noted is actually sufficient for X to contain a product subspace:

T19. *Let $C(X)$ be an algebra of continuous functions. Then X contains a product subspace if and only if $C(X)$ contains a closed ideal I and two analytic subalgebras F_1 and F_2 the closure of whose generated subalgebra is $C(X)$, and whose homomorphic images under the natural homomorphism $C(X) \rightarrow C(X)/I$ are additively related. The product subspace has a nonvacuous interior if and only if the complement of I is non-null.*

PROOF. The necessity of these conditions appears in the above comments. The sufficiency of the hypotheses will be established if we can show that $C(X)/I$ is equivalent to an algebra of continuous functions defined on a product space, for by T15, $C(X)/I$ is equivalent to $C(Y)$ where $I = I(Y)$. We shall show then that the hypotheses of T14 are satisfied for $C(X)/I$.

(a) Since F_1 and F_2 are analytic subalgebras of $C(X)$, their homo-

morphic images $H(F_1)$ and $H(F_2)$ under the natural homomorphism $H: C(X) \rightarrow C(X)/I$ are analytic subalgebras of $C(X)/I$.

(b) By hypothesis the algebras $H(F_1)$ and $H(F_2)$ are additively related.

(c) $\mathcal{C}((F_1 \cup F_2)_A) = C(X)$ implies that $(F_1 \cup F_2)_A$ is dense in $C(X)$ which implies that $H((F_1 \cup F_2)_A) = (H(F_1) \cup H(F_2))_A$ is dense in $C(X)/I$ since H is continuous and onto all of $C(X)/I$, and hence $\mathcal{C}((H(F_1) \cup H(F_2))_A) = C(X)/I$.

We have then that the hypotheses of T14 are satisfied for $C(X)/I$, and the first statement of the theorem is therefore proved. The second statement in the theorem follows from T17.

Note that T19 is a generalization of T14 in the sense that if the ideal I in T19 can be taken as the null ideal (θ), then the hypotheses of T19 are those of T14.

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