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# ON PROJECTIVE GEOMETRY OVER FULL MATRIX RINGS

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1. **Introduction.** In this note we show that projective geometry over a ring $R$ and that over the full matrix ring $R_n$ are essentially the same, and extend the fundamental theorem of projective geometry [1, p. 44] to the case of $\Phi_n$-modules, where $\Phi$ is a division ring. (By a projective geometry over $R$ we mean a lattice of all $R$-submodules of an $R$-module.) As a special case of these results we have the following: If $n \geqq 3$, any lattice isomorphism of the lattice of all left ideals of $\Phi_n$ and that of $\Psi_m$ where $\Phi$ and $\Psi$ are division rings, is induced by an isomorphism of $\Phi_n$ and $\Psi_m$. We obtain also an extension of the basis theorem for vector spaces to $\Phi_n$-modules.

Other extensions of the fundamental theorem of projective geometry have been made by Baer, for the case of $R$-modules, where $R$ is a "primary ring" in his sense [2, p. 304], and the ring of rational integers [**3**].

2. **Main theorems.** In the following, by a ring we always mean an associative ring with unit element. Let $R$ be a ring with unit

Received by the editors April 17, 1954.

element 1. An additive group $A$ is called an $R$-module if $ax$ is defined for all $a$ in $R$ and all $x$ in $A$ such that

$$\begin{aligned} a(x + y) &= ax + ay, \\ (a + b)x &= ax + bx, \\ a(bx) &= (ab)x, \\ 1x &= x \end{aligned}$$

for any $a$, $b \in R$ and any $x$, $y \in A$. A subgroup $B$ of $A$ is called an $R$-submodule if $ax \in B$ for all $a \in R$ and all $x \in B$. The set of all $R$-submodules of $A$ forms a modular lattice $L(R, A)$ with respect to group-theoretical union and intersection. We will call the lattice $L(R, A)$ a projective geometry over the ring $R$. (Usual projective geometry is the case when $R$ is a division ring.)

We denote by $R_n$ the full matrix ring of degree $n$ over $R$. Our first main theorem is the following:

THEOREM 1. *Let $R$ be a ring and $n$ be a positive integer. Then for any $R_n$-module $M$ there exists an $R$-module $A$ such that*

$$L(R_n, M) \cong L(R, A). \tag{1}$$

*Conversely, for any $R$-module $A$ there exists an $R_n$-module $M$ such that* (1) *holds.*

Now let $\mathfrak{F} = \{R, S, \cdots\}$ be a family of rings and $\pi$ be a lattice theoretical condition. We shall say that the *fundamental theorem of projective geometry* (*f.t.p.g.*) *holds in* $\mathfrak{F}$ *under* $\pi$ if, for any ring $R$ in $\mathfrak{F}$ and $R$-module $A$ such that $L(R, A)$ satisfies the condition $\pi$, the following holds:

(2) If $S$ is a ring in $\mathfrak{F}$ and if $B$ is an $S$-module such that $L(R, A) \cong L(S, B)$, then there exists an isomorphism $\sigma$ of $A$ to $B$ and an isomorphism $\sigma'$ of $R$ to $S$ such that the lattice isomorphism $A_1 \rightarrow A_1^* = B_1$, where $A_1$ denotes an arbitrary element in $L(R, A)$, is induced by $\sigma$, i.e.

$$A_1^* = \{y^\sigma \mid y \in A_1\},$$

and such that $(ax)^\sigma = a^{\sigma'}x^\sigma$ for all $a$ in $R$ and all $x$ in $A$. Our second main theorem is the following:

THEOREM 2. *Let $\mathfrak{F} = \{R, S, \cdots\}$ be a family of rings, $n$ a positive integer, and $\pi$ be a lattice-theoretic condition. If f.t.p.g. holds in $\mathfrak{F}$ under $\pi$ then f.t.p.g. also holds in $\mathfrak{F}_n = \{R_n, S_n, \cdots\}$ under $\pi$. The isomorphism $\sigma'$ of $R_n$ to $S_n$ needed in f.t.p.g. in $\mathfrak{F}_n$ can always be chosen so* *that $\sigma'$ is of the type $(a_{ij})^{\sigma'}=(a_{ij}^{\sigma''})$, where $\sigma''$ is a suitable isomorphism of $R$ to $S$.*

3. **Proof of Theorem 1.** Let $e_{ij}$ be the $n\times n$ matrix with 1 in the $i$th row and $j$th column and 0 elsewhere. It is easily seen that $e_{ii}$ commutes with any diagonal matrix. Let $a$ be an element in $R$. We shall denote by $[a]$ the diagonal matrix of which all the diagonal elements are equal to $a$. The set of all $[a]$ forms a subring of $R_n$ isomorphic to $R$.

Let $M$ be a given $R_n$-module. If we set $A=e_{11}M$, then we can consider $A$ as an $R$-module, since every $[a]$ is commutative with $e_{11}$. If $M_1$ is an $R_n$-submodule of $M$, then $A_1=e_{11}M_1$ is an $R$-submodule of $A$. We shall show that the mapping $\phi$: $M_1\rightarrow A_1=e_{11}M_1$ gives the desired isomorphism of $L(R_n, M)$ and $L(R, A)$. Let $M_1$, $M_2$ be two $R_n$-submodules of $M$ such that $e_{11}M_1\subseteq e_{11}M_2$. We shall show that $M_1\subseteq M_2$. For let $x\in M_1$. Then $e_{1i}x\in M_1$, and $e_{1i}x=e_{11}(e_{1i}x)\in e_{11}M_1$ $\subseteq e_{11}M_2\subseteq M_2$. Thus $e_{1i}x\in M_2$, and $e_{ii}x=e_{i1}(e_{1i}x)\in e_{i1}M_2\subseteq M_2$. Hence $e_{ii}x\in M_2$ for $i=1, \cdots, n$, and since $e_{11}+\cdots+e_{nn}$ is the unit element of $R_n$, $x\in M_2$. Thus $M_1\subseteq M_2$ is proved. If $e_{11}M_1=e_{11}M_2$ then clearly $M_1=M_2$. Hence we have proved that $\phi$ is univalent and preserves inclusion. Let $A_1$ be any $R$-submodule of $A$, and let $M_1$ be the set of all elements $x$ in $M$ such that $e_{1i}x\in A_1$ for all $i=1, \cdots, n$. Then $M_1$ is an $R_n$-submodule of $M$, for if $x\in M_1$ then $e_{ij}x\in M_1$ for $i, j=1, \cdots, n$ and $[a]x\in M_1$ for all $a$ in $R$, since $e_{1k}(e_{ij}x)=\delta_{ki}e_{1j}x\in A_1$, where $\delta_{ki}$ is the Kronecker delta. Since $[a]$ and $e_{1i}$ permute, and $A_1$ is an $R$-submodule of $A$, $[a]x\in M_1$, so that $M_1$ is proved to be an $R_n$-submodule of $M$. We shall show that $A_1=e_{11}M_1$. Since $e_{11}x\in A_1$ for all elements $x$ of $M_1$, we have $e_{11}M_1\subseteq A_1$. Let $x\in A_1$; then since $A_1\subseteq A=e_{11}M$, $x=e_{11}x'$ for some $x'\in M$, and $e_{11}x=x\in A_1$, $e_{1i}x=e_{1i}(e_{11}x)$ $=0$ for $i>1$. Therefore $x\in M_1$, and $x=e_{11}x\in e_{11}M_1$. Thus $A_1=e_{11}M_1$ is proved.

To prove the second part of Theorem 1 let $A$ be a given $R$-module. Let $M$ be the totality of $n$-uples $(x_1, \cdots, x_n)$ of elements of $A$. If we define addition in $M$ by adding component-wise, $M$ becomes an additive group (direct sum of $n$ copies of $A$). Let $(a_{ij})$ be an arbitrary element of $R_n$, and define $(a_{ij})(x_1, \cdots, x_n)=(y_1, \cdots, y_n)$ by $y_i$ $=a_{i1}x_1+\cdots+a_{in}x_n$ for $i=1, \cdots, n$. Then $M$ becomes an $R_n$-module, and the $R$-module $e_{11}M$ is clearly isomorphic to the $R$-module $A$. That $L(R_n, M)\cong L(R, A)$ follows as in the proof of the first half of Theorem 1, so that Theorem 1 is completely proved.

4. **Proof of Theorem 2.** Suppose that the f.t.p.g. holds in $\mathfrak{F}=\{R, S, \cdots\}$ under $\pi$, that $M$ is an $R_n$-module, and that $L(R_n, M)$

satisfies $\pi$. Let $M_1 \to M_1^* = N_1$ be a lattice isomorphism of $L(R_n, M)$ and $L(S_n, N)$, where $N$ is an $S_n$-module. By Theorem 1 there exists an $R$-module $A$ and $S$-module $B$ such that $L(R_n, M) \cong L(R, A)$, $L(S_n, N) \cong L(S, B)$. Then $L(R, A) \cong L(S, B)$ and clearly both of these lattices satisfy $\pi$. In view of the proof of Theorem 1, we may assume that $A = e_{11}M$, $B = e_{11}N$. Now by our assumption there is an isomorphism $\sigma$ of $e_{11}M$ to $e_{11}N$ and an isomorphism $\sigma'$ of $R$ to $S$ such that $e_{11}M_1^* = \{y^\sigma \mid y \in e_{11}M_1\}$ for every $R_n$-submodule $M_1$ of $M$, and such that $([a]e_{11}x)^\sigma = [a^{\sigma'}](e_{11}x)^\sigma$ for every $a$ in $R$ and every $x$ in $M$. Now define a mapping $\tau$: $x \to x^\tau$ from $M$ to $N$ by

$$x^\tau = e_{11}(e_{11}x)^\sigma + e_{21}(e_{12}x)^\sigma + \cdots + e_{n1}(e_{1n}x)^\sigma.$$

Note that $(e_{1i}x)^\sigma$ is meaningful for $x \in M$ and $i = 1, \cdots, n$, since $e_{1i}x = e_{11}(e_{1i}x) \in e_{11}M$. We shall show that $\tau$ is an isomorphism of $M$ and $N$. Since $\sigma$ is an isomorphism of $e_{11}M$ to $e_{11}N$, we have easily $(x+y)^\tau = x^\tau + y^\tau$. If $x^\tau = 0$, then $e_{1i}x^\tau = 0$ and therefore $e_{1i}e_{i1}(e_{1i}x)^\sigma = e_{11}(e_{1i}x)^\sigma = 0$. Since $(e_{1i}x)^\sigma \in e_{11}N$ we have $(e_{1i}x)^\sigma = e_{11}(e_{1i}x)^\sigma = 0$. Hence $e_{1i}x = 0$, since $\sigma$ is an isomorphism. Then $e_{ii}x = e_{i1}e_{1i}x = 0$ for $i = 1, \cdots, n$. Therefore $x = 0$. Now we have to show that for any $z \in N$ there is an $x \in M$ such that $z = x^\tau$. Since $e_{1i}z = e_{11}(e_{1i}z) \in e_{11}N$ there exists an element $x_i$ in $e_{11}M$ such that $e_{1i}z = x_i^\sigma$. Put $x = e_{11}x_1 + e_{21}x_2 + \cdots + e_{n1}x_n$. Then $e_{1i}x = e_{11}x_i = x_i$. Hence $e_{1i}z = (e_{1i}x)^\sigma$, and $e_{ii}z = e_{i1}(e_{1i}x)^\sigma$. Therefore $z = x^\tau$. Thus we have proved that $\tau$ is an isomorphism of $M$ to $N$. Now for any $a \in R$ and $x \in M$ we have

$$\begin{aligned}([a]x)^\tau &= \sum e_{i1}(e_{1i}[a]x)^\sigma = \sum e_{i1}([a]e_{1i}x)^\sigma \\ &= \sum e_{i1}[a^{\sigma'}](e_{1i}x)^\sigma = [a^{\sigma'}] \sum e_{i1}(e_{1i}x)^\sigma \\ &= [a^{\sigma'}]x^\tau,\end{aligned}$$

and $(e_{ij}x)^\tau = e_{i1}(e_{1j}x)^\sigma = e_{ij}x^\tau$. Therefore for any $(a_{ij}) \in R_n$ and $x \in M$ we have $((a_{ij})x)^\tau = (a_{ij}^{\sigma'})x^\tau$. Finally we have to show that $M_1^* = \{x^\tau \mid x \in M_1\}$ for an arbitrary $R_n$-submodule $M_1$ of $M$. We note that $y^\tau = y^\sigma$ for all $y \in e_{11}M$. Let $x$ be in $M_1$. Since we have $e_{11}M_1^* = \{y^\sigma \mid y \in e_{11}M_1\}$, and since $e_{1i}x \in e_{11}M_1$, we have $e_{1i}x^\tau = (e_{1i}x)^\tau = (e_{1i}x)^\sigma \in e_{11}M_1^*$. Hence $e_{ii}x^\tau \in e_{i1}M_1^* \subseteq M_1^*$. Therefore $x^\tau \in M_1^*$. Conversely, let $z$ be in $M_1^*$. Then we can find $x_i$ in $e_{11}M_1$ such that $e_{1i}z = x_i^\sigma = x_i^\tau$. Hence $e_{ii}z = e_{i1}x_i^\tau = (e_{i1}x_i)^\tau$ and $z = \sum (e_{i1}x_i)^\tau = (\sum e_{i1}x_i)^\tau$. Clearly $x = \sum e_{i1}x_i$ is in $M_1$, and $z = x^\tau$. Thus we have proved that the lattice isomorphism of $L(R_n, M)$ to $L(S_n, N)$ is induced by the group-isomorphism $\tau$. Theorem 2 is completely proved.

5. **Applications.** Theorem 1 shows that for any division ring $\Phi$ and any $\Phi_n$-module $M$ the lattice $L(\Phi_n, M)$ is complementary.

To move from a projective geometry over $\Phi$ to one over $\Phi_n$ more effectively, we need lattice-theoretic definitions of a few concepts in vector spaces. Let $L$ be a lattice with join $\cup$ and meet $\cap$. We shall write $X \leqq Y$ if $X \cap Y = X$. Assume that $L$ has a smallest element 0, that is, 0 is an element such that $0 \leqq X$ for all $X \in L$. A finite set $X_1, \cdots, X_r$ of elements $\neq 0$ in $L$ are said to be *independent* if $X_i \cap (X_1 \cup \cdots \cup X_{i-1} \cup X_{i+1} \cup \cdots \cup X_r) = 0$ for $i = 1, \cdots, r$. A set $\{X_\alpha\}$ of elements are said to be independent if every finite subset of $\{X_\alpha\}$ is independent. An element $X \neq 0$ in $L$ is said to be *minimal* if $X \geqq Y > 0$ implies $X = Y$. An independent set $\mathfrak{B} = \{X_\alpha\}$ of minimal elements is called a *basis* of $L$ if for every minimal element $X$ there exist finite number of elements $X_1, \cdots, X_n$ in $\mathfrak{B}$ such that $X \leqq X_1 \cup \cdots \cup X_n$. The well-known basis theorem of vector spaces [1, p. 14] can easily be seen to be equivalent to the following: If $\Phi$ is a division ring then for any $\Phi$-module $A$ the lattice $L(\Phi, A)$ has a basis, and any two bases of $L(\Phi, A)$ have the same (cardinal) number of elements. Now Theorem 1 shows that for any $\Phi_n$-module $M$ the lattice $L(\Phi_n, M)$ has a basis, and that any two bases of $L(\Phi_n, M)$ have the same number of elements. From this we may readily prove the following extension of the basis theorem of vector spaces:

*Let $\Phi$ be a division ring, $n$ be a positive integer, and $M$ be a $\Phi_n$-module. Then there exists a set $\mathfrak{B} = \{x_\alpha\}$ of elements in $M$ such that*

(i) *$\Phi_n x_\alpha$ is isomorphic to the $n$-dimensional vector space over $\Phi$ for any $x_\alpha$ in $\mathfrak{B}$,*

(ii) *any element $x$ in $M$ can be expressed as a linear combination $x = a_1 x_{\alpha_1} + \cdots + a_r x_{\alpha_r}$ with $x_{\alpha_1}, \cdots, x_{\alpha_r}$ in $\mathfrak{B}$ and $a_1, \cdots, a_r$ in $\Phi_n$,*

(iii) *$a_1 x_{\alpha_1} + \cdots + a_r x_{\alpha_r} = 0$ with $x_{\alpha_1}, \cdots, x_{\alpha_r}$ in $\mathfrak{B}$ and $a_1, \cdots, a_r$ in $\Phi_n$ implies $a_1 x_{\alpha_1} = \cdots = a_r x_{\alpha_r} = 0$. Any two sets $\mathfrak{B}$, $\mathfrak{B}'$ satisfying* (i)–(iii) *have the same number of elements.*

Theorem 2 gives us at once an extension of f.t.p.g. to the case of simple rings $\Phi_n$, since for any family $\mathfrak{F} = \{\Phi, \Psi, \cdots\}$ of division rings f.t.p.g. holds under the condition $\pi$: "there are at least three independent elements" [1, p. 44]. A similar extension when $R$ is the ring of all integers, if we use a result of Baer [3, p. 39]: *If the abelian group $G$ contains at least two independent elements of infinite order, and if $H$ is an abelian group such that the lattice of all subgroups of $G$ is isomorphic to that of $H$, then the lattice isomorphism is induced by an isomorphism of $G$ and $H$.* In this case we have to consider only torsion free modules.

Now let $R$ be a ring and $A = R \oplus \cdots \oplus R$ be a direct sum of $n$ copies of the additive group of $R$. If we define $ax = (aa_1, \cdots, aa_n)$ for $a \in R$ and $x = (a_1, \cdots, a_n) \in A$, then $A$ becomes an $R$-module.

We assume that $R$ satisfies the following two conditions:

(i) Any lattice automorphism of $L(R, A)$ is induced by some automorphism $\sigma$ of $A$ such that $(ax)^\sigma = a^{\sigma'}x^\sigma$ for every $a \in R$, $x \in A$, where $\sigma'$ is an automorphism of $R$.

(ii) Let $P$, $Q$ be in the full matrix ring $R_n$. If $PQ=1$ then $QP=1$. $Q$ will be denoted by $P^{-1}$.

THEOREM 3. *Let a ring $R$ and an integer $n$ be such that the above two conditions* (i), (ii) *are satisfied. Then any automorphism $\alpha$ of the full matrix ring $R_n$ is of the form*:

$$(a_{ij})^\alpha = U^{-1}(a_{ij}^{\alpha'})U \tag{3}$$

*where $\alpha'$ is an automorphism of $R$, and $U$ is an element of $R_n$ for which $U^{-1}$ exists.*

PROOF. We consider $R_n$ as an $R_n$-module. Then $L(R_n, R_n)$ is the lattice of all left ideals of $R_n$. Let $\mathfrak{l}$ be an arbitrary left ideal of $R_n$. If we set $\mathfrak{l}^\alpha = \{(a_{ij})^\alpha \mid (a_{ij}) \in \mathfrak{l}\}$, then $\alpha$ becomes a lattice-automorphism of $L(R_n, R_n)$. Now in view of the proof of Theorem 1, we know that $L(R_n, R_n) \cong L(R, A)$. Therefore there is a lattice-automorphism $\beta$ of $L(R, A)$ corresponding to the automorphism $\alpha$ of $L(R_n, R_n)$. By our assumption $\beta$ is induced by an automorphism $\sigma$ of $A$ such that $(ax)^\sigma = a^{\sigma'}x^\sigma$ for $a \in R$, $x \in A$, where $\sigma'$ is an automorphism of $R$. Now in view of the proof of the second part of Theorem 2 there is an automorphism $\tau$ of the additive group of $R_n$ such that $\tau$ induces the same automorphism of $L(R_n, R_n)$ as $\alpha$ and such that $((a_{ij})(b_{ij}))^\tau = (a_{ij}^{\sigma'})(b_{ij})^\tau$ for any $(a_{ij})$, $(b_{ij}) \in R_n$. We set $[1]^\tau = V$, then for any $(a_{ij}) \in R_n$ we have $(a_{ij})^\tau = (a_{ij}^{\sigma'})V$. Since $\tau$ is an automorphism of the additive group of $R_n$, there exists an element $(b_{ij})$ such that $(b_{ij})^\tau = 1$, so that if we set $W = (b_{ij}^{\sigma'})$ then $1 = WV$. Now by our assumption $[1] = VW$. Therefore if we set $(a_{ij})^\eta = W(a_{ij}^{\sigma'})V$ then $\eta$ is an automorphism of $R_n$. It is easily seen that $\eta$ induces the same lattice automorphism of $L(R_n, R_n)$ as $\tau$, and consequently, as $\alpha$.

Now we set $\iota = \alpha\eta^{-1}$. Then $\iota$ is an automorphism of $R_n$, which induces the identity lattice automorphism of $L(R_n, R_n)$. We set

$$P = \sum_{i=1}^n e_{i1}^\iota e_{1i}, \qquad Q = \sum_{i=1}^n e_{i1}e_{1i}^\iota.$$

Then $PQ = \sum e_{i1}^\iota e_{11} e_{1i}^\iota$. Since the left ideal $R_n e_{11}$ is invariant under the automorphism $\iota$, we have $e_{i1}^\iota = S_i e_{i1}$ for some $S_i \in R_n$. Then $e_{i1}^\iota e_{11} = S_i e_{i1} e_{11} = S_i e_{i1} = e_{i1}^\iota$. Therefore $PQ = \sum e_{i1}^\iota e_{1i}^\iota = \sum e_{ii}^\iota = (\sum e_{ii})^\iota = [1]$. Hence $QP = [1]$. Now we consider the automorphism $\gamma$ of $R_n$, defined by

$$X^{\gamma} = QX^{\iota}P = \sum e_{i1}e_{1i}^{\iota}X^{\iota}e_{j1}^{\iota}e_{ij}.$$

Then $e_{ij}^{\gamma}=e_{i1}e_{11}^{\iota}e_{1j}=e_{i1}e_{1j}=e_{ij}$ for any $i$, $j=1, 2, \cdots, n$. Now for any $[a]$, $[a]^{\gamma}$ commutes with all $e_{ij}^{\gamma}=e_{ij}$. Hence $[a]^{\gamma}=[b]$ for some $b\in R$. It is easily seen that $\tau$ induces an automorphism of the ring of all matrices of the form $[a]$. Therefore, there exists an automorphism $\gamma'$ of $R$ such that $[a]^{\gamma}=[a^{\gamma'}]$ for all $a\in R$. Then for any $(a_{ij})\in R_n$ we have $(a_{ij})^{\gamma}=(a_{ij}^{\gamma'})$, and $(a_{ij})^{\iota}=P(a_{ij}^{\gamma'})Q$. Therefore, from $\alpha=\iota\eta$, we have $(a_{ij})^{\alpha}=(a_{ij})^{\iota\eta}=P^{\eta}(a_{ij}^{\gamma'})^{\eta}Q^{\eta}=P^{\eta}W(a_{ij}^{\gamma'\sigma'})VQ=U^{-1}(a_{ij}^{\alpha'})U$ where we set $U=VQ^{\eta}$, $\alpha'=\gamma'\sigma'$. Thus Theorem 3 is proved.

From Baer's Theorem [3, p. 39] we know that the ring $I$ of all integers satisfies the above condition (i) for all $n$. Since $I$ can be imbedded in a field, (ii) is also satisfied for all $n$. From Theorem 3, therefore, it follows that any automorphism of the full matrix ring $I_n$ is inner.

The author wishes to express his gratitude to Professor S. A. Jennings for help and encouragement in the preparation of this work.

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