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THE SCHWARZIAN DERIVATIVE AND CONVEX FUNCTIONS

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1. **Introduction.** In a comparatively recent paper [2], Nehari has shown that if

$$f(z) = 1/z + a_1 z + a_2 z^2 + \cdots \qquad \text{for } 0 < |z| < 1 \tag{1.1}$$

and

$$|\{f(z), z\}| \leqq \frac{\pi^2}{2} \qquad \text{for } |z| < 1, \tag{1.2}$$

where $\{f(z), z\}$ is the Schwarzian derivative of $f(z)$ with respect to z, then $f(z)$ is univalent in the unit circle. The methods of Nehari can be modified to apply to functions of the form (1.1) to be shown univalent and convex in the unit circle. The principal result obtained in this paper is the following:

THEOREM 1. *If $f(z)$ is of the form* (1.1), *regular for $0<|z|<1$, and if*

$$|\{f(z), z\}| \leqq 2c_0 \qquad \textit{for } |z| < 1, \tag{1.3}$$

where c_0 is the smallest positive root of the equation

$$2x^{1/2} - \tan x^{1/2} = 0, \tag{1.4}$$

then $f(z)$ is univalent in $0<|z|<1$ and maps the interior of each circle $|z|=r<1$ onto the exterior of a convex region. The constant c_0 is the largest possible one.

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2. **Introductory topics.** Let the function $p(z)$ be analytic and single-valued for $|z|<1$. The differential equation

$$w'' + p(z)w = 0 \tag{2.1}$$

will have two linearly independent solutions $w_1(z)$ and $w_2(z)$ which are uniquely determined by the conditions

$$\begin{aligned} w_1(0) &= 1, \qquad & w_1'(0) &= 0, \\ w_2(0) &= 0, \qquad & w_2'(0) &= 1. \end{aligned} \tag{2.2}$$

These solutions are analytic and single-valued for $|z|<1$ and consequently have the following power series expansions valid for $|z|<1$

$$w_1(z) = 1 + \sum_{n=2}^{\infty} a_n z^n, \qquad w_2(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2.3}$$

For these solutions the Wronskian, $W(w_1, w_2)$, is

$$W(w_1, w_2) = \begin{vmatrix} w_1(z) & w_2(z) \\ w_1'(z) & w_2'(z) \end{vmatrix} \equiv 1 \qquad \text{for } |z| < 1. \tag{2.4}$$

It is known that for a function $f(z)$ of the form (1.1) to map the interior of each circle $|z|=r<1$ onto the exterior of a convex region, a necessary and sufficient condition is that

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \leqq 0 \qquad \text{for } |z| < 1. \tag{2.5}$$

Let $w_1(z)$ and $w_2(z)$ be the two linearly independent solutions of (2.1) subject to the conditions (2.2). Consider

$$f(z) = \frac{w_1(z)}{w_2(z)} = \frac{1}{z} + \cdots . \tag{2.6}$$

Then

$$f'(z) = \frac{w_2 w_1' - w_2' w_1}{w_2^2(z)} = -\frac{W(w_1, w_2)}{w_2^2(z)} = -\frac{1}{w_2^2(z)}, \tag{2.7}$$

$$f''(z) = \frac{2w_2'(z)}{w_2^3(z)} . \tag{2.8}$$

With these substitutions in (2.5) we have

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) \equiv 1 - 2\Re\left(\frac{zw_2'(z)}{w_2(z)}\right). \tag{2.9}$$

But condition (2.5) implies

$$\Re\left(\frac{zw_2'(z)}{w_2(z)}\right) \geqq 1/2 \qquad \text{for } |z| < 1 \tag{2.10}$$

and conversely.

The inequality (2.10) indicates that $w_2(z)$ is starlike with respect to the origin in $|z| < 1$. Since $w_2'(0) \neq 0$ it follows that $w_2(z)$ is univalent for $|z| < 1$. Thus, because $w_2(0) = 0$, it follows that $w_2(z) \neq 0$ for $0 < |z| < 1$ and $f(z)$ defined in (2.6) is analytic and single-valued in $0 < |z| < 1$ and has a simple pole at the origin.

The preceding remarks may be summarized as

THEOREM 2. *The function $f(z)$ defined in* (2.6) *will be univalent and convex for $0 < |z| < 1$ if and only if $w_2(z)$ satisfies* (2.10).

It is surprising to observe from (2.10) that the convexity of $f(z)$ depends only on $w_2(z)$. It has already been noted that $w_2(z)$ is starlike, but (2.10) indicates that $w_2(z)$ must be a member of a special class of starlike functions. An investigation of the geometric significance of this class of functions should prove of interest.

In the proof of Theorem 1, we shall make use of some relationships between the Schwarzian derivative $\{f(z), z\}$ and the solutions of the linear second order differential equation (2.1). Although these relations are, perhaps, generally known, they will be listed for convenient reference.

If $y_1(z)$ and $y_2(z)$ are any two linearly independent solutions of (2.1), the ratio

$$y_1(z)/y_2(z)$$

satisfies the differential equation

$$\{f(z), z\} = 2p(z) \tag{2.11}$$

where

$$\{f(z), z\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2. \tag{2.12}$$

Conversely any solution of (2.11) may be written as the ratio of two linearly independent solutions of (2.1). For if $w_1(z)$ and $w_2(z)$ are the two linearly independent solutions of (2.1) satisfying (2.2), then

$$\begin{aligned} y_1(z) &= c_{11}w_1(z) + c_{12}w_2(z), \\ y_2(z) &= c_{21}w_1(z) + c_{22}w_2(z), \end{aligned} \tag{2.13}$$

with

$$\begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} \neq 0 \tag{2.14}$$

and the ratio

$$\frac{y_1(z)}{y_2(z)} = \frac{c_{11}w_1(z) + c_{12}w_2(z)}{c_{21}w_1(z) + c_{22}w_2(z)} \tag{2.15}$$

involves three arbitrary constants. But the differential equation (2.11) is of third order. Hence (2.15) is a general solution and any solution of (2.11) may be obtained from (2.15) by a proper choice of constants. If $f(z)$ is to have the form (1.1), $c_{12}=c_{21}=0$ and $c_{11}=c_{22}=1$. From this it follows that

$$f(z) = w_1(z)/w_2(z). \tag{2.16}$$

Thus there will be no loss of generality if we confine our attention to the two particular solutions.

3. **The function** $c^{1/2} \cot (c^{1/2}z)$. Let c_0 be the smallest positive root of the equation (1.4). c_0 satisfies the inequalities

$$\pi/3 < 1.16 < c_0^{1/2} < 1.17 < \pi/2, \tag{3.1}$$

$$c_0 < \pi^2/4. \tag{3.2}$$

The differential equation

$$w'' + cw = 0, \qquad c > 0, \tag{3.3}$$

has the solutions

$$\begin{aligned} w_1(z) &= \cos (c^{1/2}z) = 1 + \cdots , \\ w_2(z) &= c^{-1/2} \sin (c^{1/2}z) = z + \cdots , \end{aligned} \qquad |z| < 1, \tag{3.4}$$

which satisfy conditions (2.2).

LEMMA 3.1. *Let c_0 be the smallest positive root of the equation* (1.4). *Let $f(z)=c^{1/2}$ cot $(c^{1/2}z)$, $0<c\leqq c_0$. Then $f(z)$ is univalent in $0<|z|<1$ and maps the interior of the unit circle onto the exterior of a convex region. The constant c_0 is the best possible in the sense that for $c>c_0$ there is a point z_0, with $|z_0|<1$, for which*

$$\Re\left(1 + z_0 \frac{f''(z_0)}{f'(z_0)}\right) > 0.$$

As noted in the introduction, if $f(z)$ is of the form (1.1) and satisfies

(1.2), then $f(z)$ is univalent for $0<|z|<1$. The function $f(z)$ of the lemma is the ratio $w_1(z)/w_2(z)$ of the solutions (3.4) of the differential equation (3.3). Hence from (2.11)

$$\{f(z), z\} = 2c \leqq 2c_0 \qquad \text{for } |z| < 1, \tag{3.5}$$

and from (3.2) and (1.2) we conclude that $f(z)$ is univalent for $|z|<1$.

To establish the convexity of $f(z)$ it will suffice to prove that

$$\Re\left(\frac{zw_2'(z)}{w_2(z)}\right) \equiv \Re\left(\frac{c^{1/2}z \cos (c^{1/2}z)}{\sin (c^{1/2}z)}\right) \geqq 1/2 \tag{3.6}$$

for $|z|<1$ and $0<c\leqq c_0$. If we substitute $z=x+iy$ in (3.6), simplify, and rearrange terms, we obtain

$$\begin{aligned} &\sin (c^{1/2}x) \cos (c^{1/2}x)\left[2c^{1/2}x - \tan (c^{1/2}x)\right] \\ &\qquad \geqq \sinh (c^{1/2}y) \cosh (c^{1/2}y)\left[\tanh (c^{1/2}y) - 2c^{1/2}y\right] \end{aligned} \tag{3.7}$$

for $0<c\leqq c_0$ and $x^2+y^2\leqq 1$. It will be noted that equality holds in (3.7) for $x=\pm 1$, $y=0$, and $c=c_0$. For $x^2+y^2<1$, $x\geqq 0$, $y\geqq 0$, and $0<c\leqq c_0$, the left-hand member of (3.7) is positive while the right-hand member is either negative or zero. Thus since the map of $|z|=1$ is symmetric about the real and imaginary axes, (3.6) is established and the expression $\Re(zw_2'(z)/w_2(z))$ actually attains the value $1/2$ for $x=\pm 1$, $y=0$, and $c=c_0$. For $c=c_0+\epsilon$, $\epsilon>0$ and ϵ arbitrarily small, $\Re(zw_2'(z)/w_2(z))<1/2$.

4. **Proof of Theorem 1.** To facilitate the proof of the theorem two lemmas will be introduced.

LEMMA 4.1. *If $w(z)$ satisfies* (2.1) *with $w(0)=0$ and $w'(0)=1$, then for $0<r<1$*

$$|w|^2\Re\left(\frac{zw'(z)}{w(z)}\right) \equiv r\int_0^r |w'|^2 d\rho - r\int_0^r \Re(z^2p(z))_{|z|=\rho}\frac{|w|^2}{\rho^2}\,d\rho \tag{4.1}$$

for $|z|<1$.

Let C be a rectifiable curve lying entirely within the unit circle and joining the origin to any point z within the unit circle. If (2.1) is multiplied by $[w(z)]^*$ (* denotes complex conjugate) and integrated along C, the identity known as Green's transform is obtained.

$$[[w(z)]^*\cdot w'(z)]_0^z - \int_0^z |w'|^2 dz^* + \int_0^z p(z)|w|^2 dz \equiv 0, \qquad |z| < 1. \tag{4.2}$$

In particular let C be the path from the origin to $z=re^{i\theta}$ along the ray $\theta=$ constant. If we multiply (4.2) by z and take the real part we obtain (4.1).

This particular form (4.1) of Green's transform appears to have been first used in a recent paper by Robertson [3].

LEMMA 4.2. *Let $y(\rho)$ and $y'(\rho)$ be continuous real functions of ρ for $0 \leqq \rho < 1$. For small values of ρ let $y(\rho)=O(\rho)$. Then*

$$(4.3)\quad 0 \leqq r\int_0^r [y'(\rho)]^2 d\rho - cr\int_0^r y^2(\rho)d\rho - c^{1/2}r \cot (c^{1/2}r)\cdot y^2(r)$$

for $0<r<1$ and $c>0$. Equality holds for

$$(4.4)\qquad y(\rho) = c^{-1/2}\sin (c^{1/2}\rho), \qquad c > 0.$$

To prove this lemma, consider

$$(4.5)\qquad 0 \leqq r\int_0^r [y'(\rho) - c^{1/2}\cot (c^{1/2}\rho)\cdot y(\rho)]^2 d\rho.$$

Expanding, integrating the middle term by parts, and simplifying slightly, we obtain

$$(4.6)\quad 0 \leqq r\int_0^r [y'(\rho)]^2 d\rho - c^{1/2}r[\cot (c^{1/2}\rho)\cdot y^2(\rho)]_0^r - rc\int_0^r y^2(\rho)d\rho.$$

With some further slight simplification the lemma follows.

Equality will hold in (4.3) if the integrand in (4.5) is zero. The resulting differential equation has (4.4) as a solution.

Theorem 1 now follows quite readily. The univalence of the function (1.1) is an immediate consequence of (1.3), (3.2), and the criterion of Nehari cited in the introduction.

From (2.11) and (1.3),

$$|p(z)| \leqq c_0$$

so that

$$(4.7)\qquad \Re(z^2p(z)) \leqq c_0|z|^2 \qquad \text{for } |z| < 1.$$

If $\Re(z^2p(z))$ is replaced by this bound in (4.1) we obtain

$$(4.8)\quad |w|^2\Re\left(\frac{zw'}{w}\right) \geqq r\int_0^r |w'|^2 d\rho - rc_0\int_0^r |w|^2 d\rho, \quad 0 < r < 1.$$

With $z=\rho e^{i\theta}$, let $w=u(\rho,\theta)+iv(\rho,\theta)$. Along the ray $\theta=$constant, w is a function of ρ. $u(\rho)$ satisfies the hypotheses of Lemma 4.2 as does $v(\rho)$. Substituting these functions in (4.3) we obtain the inequalities

$$
(4.9)\qquad
\begin{aligned}
0 &\leqq r\int_0^r u_\rho^2 d\rho - c_0 r\int_0^r u^2 d\rho - c_0^{1/2} r \cot (c_0^{1/2} r)\cdot u^2(r, \theta),\\
0 &\leqq r\int_0^r v_\rho^2 d\rho - c_0 r\int_0^r v^2 d\rho - c_0^{1/2} r \cot (c_0^{1/2} r) v^2(r, \theta).
\end{aligned}
$$

Adding these and simplifying we have

$$
(4.10)\quad r\int_0^r | w' |^2 d\rho - c_0 r\int_0^r | w |^2 d\rho \geqq c_0^{1/2} r \cot (c_0^{1/2} r) | w |^2.
$$

Comparing (4.8) and (4.10) we see that

$$
(4.11)\qquad \Re\left(\frac{zw'}{w}\right) \geqq c_0^{1/2} r \cot (c_0^{1/2} r) \qquad \textit{for } | z | < 1.
$$

In Lemma 3.1 it was shown that $\Re(c^{1/2}z \cot (c^{1/2}z)) \geqq 1/2$ for $0 < c \leqq c_0$ and $|z| < 1$. The particular solution $w_2(z)$ satisfies the hypothesis of Lemma 4.1 and may be substituted for $w(z)$ in (4.11). We have finally

$$
(4.12)\qquad \Re\left(\frac{zw_2'(z)}{w_2(z)}\right) \geqq 1/2 \qquad \text{for } | z | < 1.
$$

5. Some consequences of Theorem 1.

COROLLARY 5.1. *Let* c_0 *be the smallest positive root of the equation* (1.4). *Let*

$$
(5.1)\qquad F(z) = z + b_2 z^2 + \cdots
$$

be analytic in $|z| < 1$ *with*

$$
(5.2)\qquad | \{F(z), z\} | \leqq 2c_0 \qquad \textit{for } | z | < 1.
$$

Then $F(z)$ *maps the interior of the unit circle onto a region which is starlike with respect to the origin and every circle passing through the origin cuts the boundary of the region in, at most, two points.*

For the function $f(z) = 1/F(z)$ we have

$$
(5.3)\qquad \{f(z), z\} = \{F(z), z\}.
$$

From (5.2) and Theorem 1, $f(z)$ is convex. The boundary of the region mapped by $f(z)$ is cut by any straight line in, at most, two points. Under the transformation $F(z) = 1/f(z)$ the region mapped by $f(z)$ goes into the region mapped by $F(z)$ and straight lines transform into circles through the origin. Since $f(z)$ maps each circle $|z| = r < 1$ onto the exterior of a convex region, and a fortiori a starlike region,

$$\Re\left(\frac{zf'(z)}{f(z)}\right) \leqq 0 \qquad \text{for } |z| < 1. \tag{5.4}$$

But

$$\Re\left(\frac{zF'(z)}{F(z)}\right) = -\Re\left(\frac{zf'(z)}{f(z)}\right) \geqq 0 \qquad \text{for } |z| < 1 \tag{5.5}$$

and it follows that $F(z)$ is starlike with respect to the origin for $|z| < 1$.

THEOREM 3. *Let*

$$g(z) = z + b_2 z^2 + \cdots \tag{5.6}$$

be analytic for $|z| < 1$ *and real on the real axis. Let*

$$\Re(\{zg'(z), z\}) \geqq -\frac{\pi^2}{2} \qquad \text{for } |z| < 1. \tag{5.7}$$

Then $g(z)$ *is univalent in* $|z| < 1$ *and maps the interior of the unit circle into a region which is convex in the direction of the imaginary axis. The constant* $-\pi^2/2$ *is the best possible one.*

Fejér [1] has shown that if $zg'(z)$ is typically-real, then $g(z)$ is univalent and convex in the direction of the imaginary axis. The proof then reduces to showing that $zg'(z)$ is typically-real. For convenience let

$$h(z) = zg'(z) = z + \cdots, \qquad |z| < 1. \tag{5.8}$$

It follows from the definition of $g(z)$ that $h(z)$ is real for real z. For any point z_0 with $|z_0| < 1$

$$h(z_0) = [h(z_0^*)]^*. \tag{5.9}$$

If $h(z)$ is to satisfy the differential equation (2.11) then, as noted earlier, $h(z) = y_1(z)/y_2(z)$ where $y_1(z)$ and $y_2(z)$ are any two linearly independent solutions of (2.1). However, if $h(z)$ is to be of the form (5.8), the two solutions must be normalized and

$$h(z) = \frac{w_2(z)}{w_1(z)} \tag{5.10}$$

where $w_1(z)$ and $w_2(z)$ are the two solutions of (2.1) satisfying conditions (2.2).

If $h(z)$ is not typically-real there is a value z_1, with $|z_1| < 1$ and $\Im(z_1) \neq 0$, and some real number α such that

$$h(z_1) = h(z_1^*) = \alpha. \tag{5.11}$$

This is equivalent to saying that the function $w(z)$ where

$$w(z) = w_2(z) - \alpha w_1(z) \tag{5.12}$$

has zeros at z_1 and z_1^*. From this point the proof requires but a slight modification of that of Nehari's Theorem II in [2] and will not be repeated here.

The function $w(z) = e^{\pi z}$ shows that $-\pi^2/2$ is the best possible constant, where $w(z)$ denotes $1+\pi z g'(z)$.

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