ON THE RADIAL LIMITS OF ANALYTIC FUNCTIONS

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1. Examples have been given [5, p. 185] of functions $f(z)$, analytic in the unit-circle K: $|z|<1$, and not identically constant, for which the radial limit $f(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$ is zero for all $e^{i\theta}$ on $|z|=1$ except for a set of linear measure zero. In view of the Riesz-Nevanlinna theorem [6, p. 197], such functions cannot be bounded, or even of bounded characteristic, in $|z|<1$. Functions of this sort appear again whenever we have an analytic function $f(z)$ whose radial limits coincide almost everywhere with the radial limits of a bounded analytic function $g(z)$, for the difference $F(z)=f(z)-g(z)$ has a radial limit zero almost everywhere on $|z|=1$. The Riesz-Nevanlinna theorem shows that, if $f(z)$ is bounded, or of bounded characteristic, and if the radial limit values of $f(z)$ coincide almost everywhere on an arc of $|z|=1$ with the radial limit values of $g(z)$, then $F(z)$ must be identically zero in $|z|<1$. The object of this note is to discuss certain aspects of the behavior of nonconstant analytic functions whose radial limits vanish almost everywhere on an arc $A\,(\theta_1<\theta<\theta_2)$ of $|z|=1$. One result of such a study (which the author plans as a sequel to this note) will be to give some idea of the way in which a function $f(z)$, whose radial limits coincide almost everywhere with the radial limits of a function $g(z)$ of bounded characteristic, can differ from $g(z)$.

We shall say that a nonconstant function $f(z)$, analytic in $|z|<1$, is of *class* (*LP*) on an arc A of $|z|=1$, if $\lim_{r\to 1}f(re^{i\theta})=f(e^{i\theta})=0$ for almost all $e^{i\theta}$ belonging to the arc A. If the arc A is the whole circumference $|z|=1$, we shall say simply that the function $f(z)$ is of class (LP).

One property of functions which are of class (LP) on an arc A is immediate: the *cluster set* of $f(z)$ at each point $e^{i\theta_0}$ of A (i.e., the set of all values α with the property that there exists a sequence $\{z_n\}$, $|z_n|<1$, $\lim_{n\to\infty} z_n=e^{i\theta_0}$, such that $\lim_{n\to\infty} f(z_n)=\alpha$) is the whole plane. For, if there is a point $e^{i\theta_0}$ on A and a complex number α which does not belong to the cluster set of $f(z)$ at $e^{i\theta_0}$, then there is a circular neighborhood $V(e^{i\theta_0})$ of $e^{i\theta_0}$ such that, in $V(e^{i\theta_0})\cap K$, the function $g(z)=[f(z)-\alpha]^{-1}$ is analytic and bounded. Since the function $g(z)$ has the constant limit $-1/\alpha$ along almost all normal segments drawn to that arc of $|z|=1$ which bounds $V\cap K$, it follows from a simple corollary of the Riesz-Nevanlinna theorem that $g(z)$, and hence $f(z)$,

Presented to the Society, February 27, 1954; received by the editors April 6, 1954 and, in revised form, May 24, 1954.

must be identically constant in $V \cap K$ and, a fortiori, in $|z|<1$. This property, i.e., that the cluster set is the whole plane, sometimes called the Weierstrass property, suggests that we investigate the values ζ which $f(z)$ admits as *asymptotic values*, i.e., the values to which $f(z)$ tends as z approaches a point P of $|z|=1$ along a curve terminating at P. We shall show (Theorem 1) that every complex value ζ (including ∞) is an asymptotic value of a function $f(z)$ of class (LP) provided that the ζ-points, i.e., the points z_k for which $f(z_k)=\zeta$, satisfy the condition

$$\sum_{k=1}^{\infty} (1 - |z_k|) < \infty. \tag{1}$$

In Theorem 2 we show that a function of class (LP) on an arc A admits every complex number ζ as an asymptotic value in every neighborhood of every point $e^{i\theta}$ of A if the ζ-points in some neighborhood $V(e^{i\theta}) \cap K$ of $e^{i\theta}$ satisfy (1). Theorem 2 then contains Theorem 1, but the proof of Theorem 1 is considerably simpler, and we give a separate proof.

LEMMA 1. *Let $f(z)$ be analytic and different from* 0 *in* $|z|<1$, *and let the modulus* $|f(re^{i\theta})|$ *have radial limit* 1 *for almost all* $e^{i\theta}$ *on* $|z|=1$. *Then unless $f(z)$ is identically constant in* $|z|<1$, *there exists a Jordan arc* $\mathcal{L}$, *lying in* $|z|<1$ *and terminating at a point* $e^{i\theta_0}$ *of* $|z|=1$, *such that, as* $z \to e^{i\theta_0}$ *along* $\mathcal{L}$, *either* $f(z) \to 0$ *or* $f(z) \to \infty$. *If there exists no path along which* $f(z) \to 0$, *then* $|f(z)|>1$ *in* $|z|<1$.

Lemma 1 is equivalent to Theorems 5 and 6 of [3], and its proof is omitted here. For brevity, we shall say that a function which is analytic in $|z|<1$ and whose modulus $|f(re^{i\theta})|$ has radial limit 1 for almost all $e^{i\theta}$ on $|z|=1$ will be called of *class* (*U*) in $|z|<1$.

THEOREM 1. *If $f(z)$ is analytic in* $|z|<1$ *and of class* (LP), *then every complex number* (*including* ∞) *which satisfies* (1) *is an asymptotic value of* $f(z)$.

Assume that a finite ζ satisfying (1) is not an asymptotic value of $f(z)$; clearly, we need not consider the case that $\zeta=0$. Since $f(z)$ is of class (LP), the function $\phi(z)=\zeta^{-1}[\zeta-f(z)]$ has radial limit 1 almost everywhere and is then of class (U) in $|z|<1$. Because the ζ-points of $f(z)$ satisfy (1), we may write $\phi(z)=B_\zeta(z)F(z)$, where $B_\zeta(z)$ is a Blaschke product extended over the zeros of $\phi(z)$. It is well known [7, p. 94] that the radial limits of a Blaschke product exist and have modulus 1 almost everywhere on $|z|=1$. From this it follows that $F(z)$ is of class (U) without zeros in $|z|<1$. It is then a consequence of Lemma 1 that, unless $F(z)$ is identically constant, $F(z)$ must admit either 0 or ∞ as an aymptotic value. We remark first that $F(z)$ cannot be constant; for if $\phi(z)$ reduces to a Blaschke product whose radial limit is 1 almost everywhere, the Riesz-Nevanlinna theorem shows that $\phi(z)$, and consequently $f(z)$, is constant. We assert next that 0 must be an asymptotic value of $F(z)$; otherwise $|F(z)|>1$ in $|z|<1$, so that $\phi(z)$ could be expressed as the quotient of two bounded functions in $|z|<1$, i.e., $\phi(z)$ would be of bounded characteristic in $|z|<1$. Again, by the Riesz-Nevanlinna theorem, $\phi(z)$ would be constant in $|z|<1$. Since zero must now be an asymptotic value of $F(z)$, and since $B_\zeta(z)$ is bounded, $\phi(z)$ must admit zero as an asymptotic value, so that ζ is an asymptotic value of $f(z)$.

To show that $f(z)$ admits ∞ as an asymptotic value,[1] we remark that the function $g(z)=e^{f(z)}$ is of class (U) without zeros in $|z|<1$. Applying Lemma 1 to $g(z)$, we see that, since $g(z)$ is not constant, $g(z)$ admits either 0 or ∞ as an asymptotic value, so that there exists at least one path $\mathcal{L}$ terminating at some point $e^{i\theta_0}$ of $|z|=1$ along which $f(z)\to\infty$. Thus Theorem 1 is proved.

We remark that Theorem 1 is related to a recent result of Cartwright and Collingwood [2, p. 112], the added hypothesis that $f(z)$ be of class (LP) in $|z|<1$ allowing us to obtain a stronger conclusion to part of Theorem 9 of their paper.

2. In order to determine how frequently a function of class (LP) admits as an asymptotic value a complex number ζ satisfying (1), it will be necessary to use a form of the Schwarz reflection principle developed recently in [4]. We summarize this principle as a lemma.

LEMMA 2. *Let $f(z)$ be meromorphic in $|z|<1$ and let A be the arc $0\leqq\theta_1<\theta<\theta_2<2\pi$. Let there exist an $\epsilon>0$ such that $f(z)$ has no zeros or poles in the region $0<1-|z|<\epsilon$, $\theta_1<\arg z<\theta_2$, and let the modulus $|f(re^{i\theta})|$ have radial limit 1 for almost all $e^{i\theta}$ on A. Then a necessary and sufficient condition that $f(z)$ may be continued analytically across the arc A by means of the reflection principle $f(\bar{z})=1/\bar{f}(1/z)$ is that $f(z)$ admit neither 0 nor ∞ as an asymptotic value on A.*

We proceed now to the principal result of this paper.

THEOREM 2. *Let $f(z)$ be analytic in $|z|<1$ and of class* (LP) *on an arc $\alpha<\theta<\beta$ of $|z|=1$. Let A be an arbitrary sub-arc of (α, β) and ζ an arbitrary complex number (including ∞). If there is a neighborhood*

[1] The method of proof of the previous paragraph shows also that ∞ is an asymptotic value of $F(z)$, but the presence of the factor $B_\zeta(z)$ precludes an immediate inference that ∞ is an asymptotic value of $\phi(z)$, and, consequently, of $f(z)$.

$V(e^{i\theta_A})\cap K$ of the midpoint $e^{i\theta_A}$ of A in which the ζ-points of $f(z)$ satisfy (1), then there exists a point $e^{i\theta_0}$ on that part of A which bounds $V\cap K$, and a Jordan arc $\mathcal{L}$ of $|z|<1$ terminating at $e^{i\theta_0}$ such that $f(z)\to\zeta$ as $z\to e^{i\theta_0}$ along $\mathcal{L}$.

Let us suppose that there exists an arc A $(\theta_1<\theta<\theta_2)$, with midpoint $e^{i\theta_A}$ and contained in (α, β), and a complex number ζ satisfying (1) in some neighborhood $V(e^{i\theta_A})\cap K$ which $f(z)$ does not admit as an asymptotic value on that subarc B of A which bounds $V(e^{i\theta_A})\cap K$. The case $\zeta=0$ being trivial, we may suppose that ζ is not 0, and, for the moment, not ∞. Since $f(z)$ is of class (LP) on A, the function $\phi(z)=\zeta^{-1}[\zeta-f(z)]$ is analytic in $|z|<1$ and has radial limit 1 for almost all $e^{i\theta}$ on A. Since the ζ-points of $f(z)$ satisfy (1) in $V(e^{i\theta_A})\cap K$, we may write, as before, $\phi(z)=B_\zeta(z)F(z)$, where $B_\zeta(z)$ is a Blaschke product extended over the zeros of $\phi(z)$ in $V(e^{i\theta_A})\cap K$, and where $F(z)$ is analytic without zeros in $V(e^{i\theta_A})\cap K$. The function $F(z)$ must possess radial limit values of modulus 1 for almost all $e^{i\theta}$ on B. It cannot happen that $F(z)$ is the quotient of two bounded functions $V(e^{i\theta_A})\cap K$; for otherwise the Riesz-Nevanlinna theorem would imply that $\phi(z)$, and consequently $f(z)$, is identically constant. Furthermore, it is clear that no point of B can be a regular point of $F(z)$. It follows from Lemma 2 that the set of singularities of $F(z)$ on B (namely, all points of B) is the closure of the set of points $e^{i\theta}$ on B which are the terminal points of Jordan arcs along which either $F(z)\to 0$ or $F(z)\to\infty$. A simple modification of a result of Carathéodory [**1**, pp. 266–267] and the author [**3**, p. 251] shows that, unless $|F(z)|>1$ in $V(e^{i\theta_A})\cap K$, $F(z)$ must admit zero as an asymptotic value on B. Now if $|F(z)|>1$ in $V(e^{i\theta_A})\cap K$, then $\phi(z)$ is the quotient of two bounded functions in that region and, according to the corollary of the Riesz-Nevanlinna theorem, must be identically constant. Since $F(z)$ must admit 0 as an asymptotic value on B, the boundedness of $B_\zeta(z)$ implies that 0 is an asymptotic value of $\phi(z)$ on B, so that ζ is an asymptotic value of $f(z)$ on B. This contradiction proves Theorem 2 for the case that $|\zeta|<\infty$. For the case that $\zeta=\infty$, we apply Lemma 2 directly to the function $g(z)=e^{f(z)}$, thus completing the proof of Theorem 2.

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MULTIPLICATIVE GROUPS OF ANALYTIC FUNCTIONS

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Let D be a *proper* subdomain of the Riemann sphere, and let $M(D)$ be the multiplicative group of all regular single-valued analytic functions on D which have no zeros in D. It is known [1] that the algebraic structure of the ring $R(D)$ of all regular single-valued analytic functions on D determines (and is determined by) the conformal type of D. In this paper we ask the question: what information about D does the algebraic structure of $M(D)$ give, and, conversely, which properties of D determine the algebraic structure of $M(D)$? The answer is, briefly, that *$M(D_1)$ and $M(D_2)$ are isomorphic if and only if D_1 and D_2 have the same connectivity.*

Here the connectivity of D is k if the complement of D has k components, and is ∞ if the complement of D has infinitely many (countable or power of the continuum) components. The structure of $M(D)$ is described in more detail in the theorem below.

If we associate with each $f \in M(D)$ the function $g = f/|f|$ we obtain a subgroup (isomorphic to $M(D)$) of the multiplicative group $C(D)$ of all continuous functions from D into the unit circumference. Such functions have been studied in great detail by Eilenberg [2]. It is worth noting that our theorem is valid if we replace $M(D)$ by $C(D)$, and that the proof is essentially the same; but it seems more interesting to stay within the smaller group.

Before stating the theorem, it is convenient to define two subgroups of $M(D)$.

(1) Fix a point $z_0 \in D$ and let $G(D)$ be the set of all $f \in M(D)$ such that $f(z_0) = 1$. Then $M(D)$ is the direct product of $G(D)$ and the multiplicative group of the nonzero complex numbers, and $G(D)$

Presented to the Society, April 24, 1954; received by the editors March 23, 1954.