THE LOOMAN-MENCHOFF THEOREM AND SOME SUBHARMONIC FUNCTION ANALOGUES[1]

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1. Introduction. Starting with the work of Goursat [7] in 1900 there has been considerable interest in the following question: given a continuous complex function f on an open set Ω, how weak can one make the further conditions imposed to guarantee analyticity of f? In particular, conditions have been sought which restrict the behavior of f on as small as possible a subset of Ω. Theorems of this sort are frequently associated with the names of Looman and Menchoff, who were the first to show [16, p. 199] that for f sufficiently well behaved the Cauchy-Riemann equations need only be required to hold almost everywhere.

In the sequel it will be convenient for purposes of comparison to have at our disposal two theorems of Besicovitch [3], and we state them here with the same numbering.

THEOREM 1. *Let f be a bounded complex function on $\Omega - E$, where Ω is an open set and E a subset having zero length.[2] If f admits a finite derivative on $\Omega - E$, then f can be extended so as to be analytic on Ω.[3]*

THEOREM 2. *Let f be a continuous complex function on an open set Ω, and let E be the union of countably many subsets of finite length. If f has a finite derivative on $\Omega - E$, then f is analytic on Ω.*

With reference to a point raised by Saks [16, p. 201], Maker has applied Theorem 2 to prove [8, p. 267] the following generalization of the Looman-Menchoff theorem.

THEOREM 3. *Let $f = u + iv$ be a continuous complex function on an open set Ω, and let E be the union of countably many sets of finite length closed in Ω. If u and v have finite partial Dini derivates on $\Omega - E$ and satisfy the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ almost everywhere, then f is analytic on Ω.*

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[2] Length = Carathéodory linear measure.

[3] Actually, the statement of the theorem according to Besicovitch presumes f to be defined on Ω, but the present version follows at once by extending f as in the proof of Theorem 4.

Our principal aim in the present paper is to extend the Looman-Menchoff theorem to subharmonic and δ-subharmonic functions, and to employ the Looman-Menchoff techniques in determining conditions sufficient for absolute continuity of the mass distributions. Throughout the paper we shall consider only functions defined on an arbitrary open set Ω in the plane, but it is clear that the theorems involving δ-subharmonic functions extend easily to Euclidean spaces of arbitrary dimension.

To obtain a characterization of continuous subharmonic functions (Theorem 6), we employ the operators of Blaschke and Privaloff, and this suggests the possibility of a corresponding condition for analyticity. Exploitation of this approach by means of techniques due to Meier [9] results in the fact that the Cauchy-Riemann equations in Theorem 3 can be replaced by the expression

$$\liminf_{r\to 0} \frac{1}{r^2}\left|\int_{C_r(z)} f(\zeta)d\zeta\right| = 0,$$

where $C_r(z)$ denotes the positively oriented boundary of the circle of radius r about z. A similar condition in terms of a square of side r has been given by Wolff [16, p. 196], but there does not seem to be any intrinsic connection between the methods.[4]

In the majority of the applications of the Looman-Menchoff techniques continuity plays an essential role in establishing closedness of certain sets (the sets F_n in our notation). Nevertheless, it is possible to relinquish continuity under favorable conditions, such as those arising in Theorem 7. This is the case also in the following theorem, which bears the same relationship to Theorem 1 as Maker's theorem does to Theorem 2.

THEOREM 4. *Let $f=u+iv$ be a bounded complex function on $\Omega-E$, where Ω is an open set and E is the union of countably many sets of zero length closed in Ω. If*

$$\limsup_{\zeta\to z}\left|\frac{f(\zeta)-f(z)}{\zeta-z}\right| < +\infty \qquad for\ z\ on\ \Omega-E$$

and the Cauchy-Riemann equations $u_x=v_y$, $u_y=-v_x$ hold almost everywhere on $\Omega-E$, then f can be extended so as to be analytic on Ω.

A proof of Theorem 4 will be given later in §4.

In the δ-subharmonic case the exceptional set E is taken as an

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[4] In dealing with rectangles in situations of this sort, essential use is made of their subdivision properties.

arbitrary Borel F_σ set of capacity zero (see [6, Chap. VII]). Since the irregular boundary points of any open set comprise an F_σ of capacity zero, it seems likely that the theorems (or at least the methods) indicated here may have applications related to the Dirichlet problem. In fact, we are led to conjecture that the Looman-Menchoff techniques should be of value in a wide class of problems involving the approximation of linear operators by difference operators.[5]

2. Subharmonic function theorems of Looman-Menchoff type.

That harmonic functions admit a Looman-Menchoff characterization has been shown by Beckenbach [2], under the assumption that the exceptional set E is countable.

We observe that modifications of the sort made by Maker [8, p. 266] in the proof of the Looman-Menchoff theorem can also be made in the proof of the theorem of Beckenbach, but it is just as easy to establish a more inclusive condition sufficient for subharmonicity.

THEOREM 5. *Let u be a real-valued function having continuous first partial derivatives on an open set Ω, and let E be a Borel F_σ set of capacity zero. If the first partial Dini derivates of u_x and u_y are finite on $\Omega - E$ and $\Delta u \equiv u_{xx} + u_{yy} \geqq 0$ holds almost everywhere, then u is subharmonic on Ω.*

Like all of the currently known proofs of the Looman-Menchoff theorem and its generalizations, our proof of Theorem 5 leans heavily on the Baire category theorem.

Let F be the set of all points $z(\in\Omega)$ such that u is not subharmonic on any neighborhood of z. Clearly F is closed in Ω and u is subharmonic on $\Omega - F$. Suppose that $F \neq \varnothing$.

For positive integral n let us define F_n as the set of all points $z(\in\Omega)$ such that $|h| < 1/n$ (h being real) implies

$$| u_x(z + h) - u_x(z) | + | u_x(z + ih) - u_x(z) |$$
$$+ | u_y(z + h) - u_y(z) | + | u_y(z + ih) - u_y(z) | \leqq n | h |.$$

The sets F_n are closed in Ω and in their aggregate cover $\Omega - E$. Further, we have $E = \bigcup_{n=1}^{\infty} E_n$, where each E_n is a set of capacity zero closed in Ω. Since the countable family of relatively closed sets F_n and E_n covers Ω, and a fortiori F, the Baire category theorem ensures the existence of a neighborhood ω such that $F \cap \omega$ is nonempty and lies in one of the covering sets.

The possibility of $F \cap \omega$ being contained in some E_n is, however,

ruled out by a theorem of Brelot [4, p. 31] which asserts that a function bounded above on a region and subharmonic outside of a relatively closed subset of capacity zero admits a unique subharmonic extension on the region[6] (by continuity this extension must in our case coincide with u). There is thus an index N for which $F \cap \omega \subset F_N$.

A proof of the following lemma is given in the monograph of Menchoff [10, p. 10].[7] Let w be a continuous real-valued function on a square $Q = [\alpha, \alpha+s] \times [\beta, \beta+s]$, and let $F(\neq \varnothing)$ be a closed subset of Q and M a constant such that $(x, y) \in F$ and $(x+h, y)$, $(x, y+k) \in Q$ imply

$$\left| w(x + h, y) - w(x, y) \right| \leqq M \left| h \right|, \quad \left| w(x, y + k) - w(x, y) \right| \leqq M \left| k \right|.$$

If $R = [x_1, x_2] \times [y_1, y_2]$ is the smallest (possibly degenerate) rectangle having edges parallel to the coordinate axes and containing F, then

$$\left| \int_{x_1}^{x_2} [w(x, y_2) - w(x, y_1)] dx - \int_F w_y da \right| \leqq 5Ma(R - F),$$

$$\left| \int_{y_1}^{y_2} [w(x_2, y) - w(x_1, y)] dy - \int_F w_x da \right| \leqq 5Ma(R - F),$$

where a denotes Lebesgue plane measure.[8]

Beckenbach [2] has shown how an argument paralleling that used by Menchoff in the analytic case leads at once to

$$\int_{\partial Q} (\partial u/\partial \nu) ds = \int_{F \cap Q} \Delta u da \geqq 0$$

(ν denoting the outward normal and ∂Q the positively oriented boundary of Q) for every square $Q \subset \omega$. It is then obvious from the proof of Lemma 3 of Beckenbach [2] that this makes u subharmonic on ω. The resulting contradiction forces $F = \varnothing$, completing the proof.

Replacing the inequality in Theorem 5 by equality plainly results in harmonicity of u and shows that the exceptional set in the theorem of Beckenbach can be allowed to be an F_σ of capacity zero.

We recall [1, p. 335] that for u an integrable function the *Blaschke difference operator* Δ_r^B is defined by

$$\Delta_r^B u(z) = (4/r^2) [\mu_r u(z) - u(z)],$$

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[6] Although stated in terms of a closed set by Brelot, it is evident that the set need only be relatively closed.

[7] Note that the corresponding lemma in Saks [16, p. 198] and in Beckenbach [2] is unnecessarily restrictive.

[8] The Lipschitz conditions on w imply that w_x and w_y exist almost everywhere and are summable on F [16, Chap. V].

where $\mu_r u(z)$ is the integral mean of u over the circumference of the circle of radius r about z. Denoting by $\overline{\Delta}^B u(z)$ and $\underline{\Delta}^B u(z)$, respectively, the limit superior and limit inferior of $\Delta_r^B u(z)$ as $r \to 0$, we obtain the *upper* and *lower Blaschke operators*, $\overline{\Delta}^B$ and $\underline{\Delta}^B$. Use of the areal mean $\alpha_r u(z)$ in place of the circumferential mean $\mu_r u(z)$, and the factor $8/r^2$ in place of $4/r^2$, serves to define the corresponding *Privaloff operators* Δ_r^P, $\overline{\Delta}^P$, and $\underline{\Delta}^P$ analogously.

As we proceed to show, a Looman-Menchoff criterion for continuous functions to be subharmonic can be given in terms of the operators of Blaschke and Privaloff.[9]

THEOREM 6. *Let u be a continuous real-valued function on an open set Ω. Then for u to be subharmonic on Ω it is necessary and sufficient that $\underline{\Delta}^B u > -\infty$ hold except perhaps on a Borel F_σ set of capacity zero and that $\underline{\Delta}^B u \geqq 0$ hold almost everywhere.*

PROOF. The necessity is obvious. For the sufficiency we take F as the set of all points $z(\in \Omega)$ such that u is not subharmonic on any neighborhood of z, and define F_n as the set of all points $z(\in \Omega)$ such that $0 < r < 1/n$ implies $\Delta_r^B u(z) \geqq -n$. These sets are closed in Ω by virtue of the continuity of $\Delta_r^B u$, so that if F is not empty, we conclude exactly as in the proof of Theorem 5 that $F_N \supset F \cap \omega \neq \varnothing$ for some index N and some neighborhood ω.

Now, from $\underline{\Delta}^B u \geqq -N$ on ω, it follows[10] that the function defined by $u(x, y) + Nx^2/2$ is subharmonic on ω, and there results $\Delta_r^B u \geqq -N$ on the neighborhood obtained by shrinking ω radially through a distance r. It is evident also that the areal mean operator α_ρ commutes with the Blaschke difference operator to yield $\Delta_r^B \alpha_\rho u = \alpha_\rho \Delta_r^B u$. Thus, taking advantage of the fact that the functions $\Delta_r^B u$ are bounded below, we can apply Fatou's lemma to infer

$$\underline{\Delta}^B \alpha_\rho u \geqq \alpha_\rho \underline{\Delta}^B u \geqq 0.$$

This implies [13, p. 14] that the continuous functions $\alpha_\rho u$ are subharmonic. Since they converge uniformly to u on compact subsets, u must be subharmonic on ω, contradicting our assumption of $F \neq \varnothing$.

Observe that the hypothesis of continuity on u cannot be supplanted by even such a plausible condition as $\lim \sup_{\zeta \to z} u(\zeta) = u(z)$ for all $z \in \Omega$. This follows from the fact that there exist nonempty perfect sets of capacity zero [11, p. 153] (the function u defined as 1 on such a set and 0 elsewhere is not subharmonic).

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[9] We carry out the details only for the case of the Blaschke operator, but the Privaloff operator behaves in the same way.

[10] See, for example, the proof of Theorem 19 of [1].

It is of interest to compare Theorem 6 with the following theorem of Privaloff [12]: a necessary and sufficient condition for a continuous real-valued function u to be subharmonic on an open set Ω is that $\overline{\Delta}^P u > -\infty$ hold except possibly on a closed set of capacity zero and $\overline{\Delta}^P u \geq 0$ hold almost everywhere. To allow the exceptional set to be an F_σ, we have thus had to pay only the rather reasonable price of replacing the upper by the lower operator.

An immediate consequence of Theorem 6 is

COROLLARY 6.1. *A real-valued function h on an open set Ω is harmonic if and only if h is continuous, $\lim \sup_{r \to 0} |\Delta_r^B h|$ is finite except perhaps on a Borel F_σ set of capacity zero, and $\Delta^B h = 0$ almost everywhere.*

For δ-subharmonic generalizations of Theorems 5 and 6 we have recourse to an elementary argument of the kind used in proving Theorem 19 of [1]. It follows that if, instead of requiring the operators appearing in these theorems to be non-negative almost everywhere, we require them to be locally essentially bounded below, then the function u is δ-subharmonic on Ω.

3. **Absolute continuity of the mass distributions.** Aside from yielding conditions for subharmonicity and δ-subharmonicity, the Looman-Menchoff techniques can be applied to the problem of determining when the mass distribution for a subharmonic or δ-subharmonic function is absolutely continuous (that is, is given as the Lebesgue integral of a summable density function).

In developing some sufficient conditions of this sort, we shall have use for two preliminary lemmas. The first of these lemmas is a special case of Theorem I of Rudin [15], but we present a brief independent proof based on Rosenbloom's characterization of the mass distribution for a potential [14].[11] It will be convenient to have on hand the following notational conventions: (1) for N any real number, λ_N will signify the function defined on the plane by

$$\lambda_N(x, y) = Nx^2/2;$$

(2) for Ω an open set and r a positive number, Ω_r will signify the set of all points of Ω whose distance from the boundary exceeds r.

LEMMA 1. *Let w be a real-valued function continuous on an open set Ω. If $\underline{\Delta}^B w$ is bounded above and $\overline{\Delta}^B w$ is bounded below, then w is δ-subharmonic and its mass distribution is absolutely continuous.*

PROOF. Choosing N such that $\underline{\Delta}^B w \leq N$ and $\overline{\Delta}^B w \geq -N$, we see

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[11] See also Theorem 13 of [1].

that $\lambda_N - w$ and $\lambda_N + w$ are subharmonic. There results the bounded-
ness condition

$$- N \leqq \overset{B}{\Delta_r} w \leqq N$$

on Ω_r. Since w is δ-subharmonic, a lemma of Rudin [15, p. 280] assures
us that $\Delta^B w$ exists almost everywhere and is summable on compact
subsets of Ω. Hence, if m is the mass distribution for w, the theorem of
Rosenbloom mentioned above yields

$$- m(e) = \lim_{r \to 0} \frac{1}{2\pi} \int_e \overset{B}{\Delta_r} w \, da = \frac{1}{2\pi} \int_e \overset{B}{\Delta} w \, da$$

for all bounded Borel sets e with closure in Ω and having no mass on
their boundaries. The lemma is thus established.

The second lemma is readily reduced to the preceding one.

LEMMA 2. *Let* Ω *be an open set,* F *a subset closed in* Ω, *and* U *the
potential of a negative mass distribution* m *on* F. *If for some constant
N and all sufficiently small $r(>0)$ the inequality*

$$\overset{B}{\Delta_r} U \leqq N$$

*holds on $F \cap \Omega_r$, then U is continuous on Ω and m is absolutely con-
tinuous.*

PROOF. Taking z and z_0 as points of F, it being assumed further that
z_0 is a limit point, we have for small r

$$| U(z) - U(z_0) | \leqq | U(z) - \mu_r U(z) | + | \mu_r U(z) - \mu_r U(z_0) |$$
$$+ | \mu_r U(z_0) - U(z_0) |$$
$$\leqq | \mu_r U(z) - \mu_r U(z_0) | + Nr^2/2 \xrightarrow[z \to z_0]{} Nr^2/2$$

by virtue of the continuity of $\mu_r U$. Hence, U is continuous as a func-
tion on F, and it follows from the work of Evans [5, p. 238] that U
must be continuous on Ω. An application of Lemma 1 completes the
proof.

With this information at our disposal we are in a position to prove
a theorem related to Theorem II of Rudin [15]. The distinction
here is similar to that between Theorem 6 and the theorem of Privaloff
noted in connection with it: we replace the lower by the upper
operator but allow the exceptional set to be an F_σ. However, in the
present case we receive the unexpected bonus of not having to re-
quire continuity.

THEOREM 7. *Let Ω be an open set and u a subharmonic function on Ω having the property that the set $\{z\colon u(z) = -\infty\}$ supports none of the mass of u. If $\overline{\Delta}^B u < +\infty$, except perhaps on a Borel F_σ set of capacity zero, then the mass distribution for u is absolutely continuous and there exists a subset E of capacity zero closed in Ω such that $\overline{\Delta}^B u < +\infty$ on $\Omega - E$.*

PROOF. We take F as the set of all points $z(\in\Omega)$ such that on no neighborhood ν of z does u satisfy the following conditions: (i) the mass distribution m for u is absolutely continuous on ν, and (ii) for some subset A_ν of capacity zero closed in ν we have $\overline{\Delta}^B u < +\infty$ on $\nu - A_\nu$. It is obvious that F is closed in Ω. Moreover, u possesses properties (i) and (ii) on the open set $\nu = \Omega - F$, as is easily seen by paracompactness considerations. Let us suppose that F is nonempty.

For $n = 1, 2, \cdots$ we denote by F_n the set of all points $z(\in\Omega)$ such that $0 < r < 1/n$ implies $\Delta_r^B u(z) \leq n$. Invoking the continuity of $\mu_r u$ and the fact that $\lim\sup_{z\to z_0} u(z) = u(z_0)$, we find that

$$\Delta_r^B u(z_0) = \lim_{z\to z_0}\inf \Delta_r^B u(z) \leq \lim_{z(\in F_n)\to z_0}\inf \Delta_r^B u(z) \leq n$$

for $0 < r < 1/n$ and z_0 a limit point of F_n. The sets F_n are thus closed in Ω.

It is evident that Ω is covered by the sets F_n and E_n $(n = 1, 2, \cdots)$, the latter being suitably chosen sets of capacity zero closed in Ω. Hence, there exists a neighborhood $\omega(\subset\Omega)$ such that $F\cap\omega$ is nonempty and lies in one of the covering sets. This cannot occur, however, for any E_n, in view of our hypothesis prohibiting sets of capacity zero from supporting any of the mass of u. It follows that $F\cap\omega\subset F_N$ for some index N, so that

$$\Delta_r^B u(z) \leq N$$

for z on $F\cap\omega$ and $0 < r < 1/N$.

We observe that this inequality remains valid for the potential U obtained by restricting m to $F\cap\omega$, inasmuch as u is the sum of U and a subharmonic function. Lemma 2 now applies to U, establishing absolute continuity of m on $F\cap\omega$, and the resulting contradiction shows that F is empty.

Combining this result with Theorem 6 gives rise to a direct extension to the δ-subharmonic case.

COROLLARY 7.1. *Let w be a real-valued function continuous on an open set Ω. If $\overline{\Delta}^B w$ and $\underline{\Delta}^B w$ are finite, except perhaps on a Borel F_σ set*

*of capacity zero, and if $\underline{\Delta}^B w$ is locally essentially bounded below, then w
can be represented as the difference of two continuous subharmonic func-
tions having absolutely continuous mass distributions.*

The proof is based on consideration of the function $w - \lambda_c$ on a
neighborhood over which c is an essential lower bound for $\underline{\Delta}^B w$.
Theorem 6 shows $w - \lambda_c$ to be subharmonic, and Theorem 7 shows its
mass distribution to be absolutely continuous.

We then make use of the following generalization of Poisson's equa-
tion, given by Rudin [15, 3.6]: if the mass distribution for a δ-sub-
harmonic function w is absolutely continuous and has density ρ, then
$\Delta^B w = -2\pi\rho$ almost everywhere. Since the potential of a mass dis-
tribution defined by a bounded density function is continuous, it is
apparent by splitting $\Delta^B w / 2\pi$ into its positive and negative parts
that the function w in Corollary 7.1 can be represented as the dif-
ference of two continuous subharmonic functions having absolutely
continuous mass distributions.

4. The analytic case. Proof of Theorem 4. Let us first of all verify
the assertion made in §1 that Theorem 3 remains in force when the
Cauchy-Riemann equations are replaced by the equation

$$\liminf_{r \to 0} \frac{1}{r^2} \left| \int_{C_r(z)} f(\zeta)d\zeta \right| = 0.$$

We take F as the set of all points $z (\in \Omega)$ such that f is not analytic
on any neighborhood of z, and for positive integral n define F_n as
the set of all points $z (\in \Omega)$ such that $|h| < 1/n$ (h real) implies

$$|f(z+h) - f(z)| + |f(z+ih) - f(z)| \leq n|h|.$$

Under the assumption that F is nonempty the application of Besico-
vitch's theorem employed by Maker [8, p. 267] serves to establish
the existence of a neighborhood ω and an index N such that $F_N \supset F$
$\cap \omega \neq \varnothing$.

From a result of Meier [9, p. 186] it is evident that ω can be chosen
in such a way that f satisfies a Lipschitz condition

$$|f(\zeta) - f(z)| \leq M|\zeta - z|$$

for ζ, z on ω. Also, the continuity of u implies that the areal mean
function $\alpha_r u$ is continuously differentiable and in fact [13, p. 11] has
the partial derivatives

$$\frac{\partial \alpha_r u(z)}{\partial x} = \frac{1}{\pi r} \int_0^{2\pi} u(z + re^{i\theta}) \cos\theta d\theta,$$

$$\frac{\partial \alpha_r u(z)}{\partial y} = \frac{1}{\pi r} \int_0^{2\pi} u(z + re^{i\theta}) \sin \theta d\theta.$$

Since the Lipschitz condition guarantees the existence almost everywhere of u_x, u_y [16, Chap. V] and justifies differentiating under the integral sign, we have access to the further identities

$$(\alpha_r u)_x = \alpha_r(u_x) \quad \text{and} \quad (\alpha_r u)_y = \alpha_r(u_y).$$

Similar expressions hold for v, and there results

$$\frac{1}{\pi r^2} \int_{C_r(z)} f(\zeta) d\zeta = i[\alpha_r u_x(z) - \alpha_r v_y(z)] - [\alpha_r u_y(z) + \alpha_r v_x(z)].$$

In the limit as $r \to 0$ this yields $u_x = v_y$ and $u_y = -v_x$ almost everywhere on ω. Hence, from Theorem 3 (or the observation that the analytic functions $\alpha_r u + i\alpha_r v$ converge uniformly on compact subsets to $u + iv$) it is apparent that f is analytic on ω, so that $F = \varnothing$.

This completes the proof of our assertion, and we turn to the demonstration of Theorem 4.

Here the function f is given only on $\Omega - E$, but we immediately extend it to all of Ω in the following way. With each point z of E we associate arbitrarily a sequence $\{\zeta_k(z)\}$ of points of $\Omega - E$ such that $\lim_{k \to \infty} \zeta_k(z) = z$ and $\{f[\zeta_k(z)]\}$ is convergent; this is clearly possible since f is bounded. Then $f(z)$ is specified as the limit of $\{f[\zeta_k(z)]\}$.

Having thus defined f on Ω, we let F be the set (closed in Ω) consisting of all points $z(\in \Omega)$ for which f is not analytic on any neighborhood of z. As in the preceding proofs we assume that F is not empty and from this trace a contradiction.

For positive integral n, F_n will denote the set of all points $z(\in \Omega)$ such that $|\zeta - z| < 1/n$ implies

(*) $$|f(\zeta) - f(z)| \leq n|\zeta - z|.$$

The following direct argument shows that each F_n is closed in Ω. Take z_0 as a limit point of F_n, ζ as a point of Ω such that $|\zeta - z_0| < 1/n$, ϵ as the positive number $1/n - |\zeta - z_0|$, and $\{z_k\}$ as any sequence of points of F_n converging to z_0 and satisfying $|z_k - z_0| < \epsilon$. Since $|\zeta - z_k| < 1/n$ and $|z_k - z_0| < 1/n$, we find

$$|f(\zeta) - f(z_0)| \leq |f(\zeta) - f(z_k)| + |f(z_0) - f(z_k)|$$
$$\leq n(|\zeta - z_k| + |z_0 - z_k|) \xrightarrow[k \to \infty]{} n|\zeta - z_0|.$$

Moreover, the method used in extending f ensures that the sets F_n cover $\Omega - E$. That is, the inequality (*) holds at each point z of

$\Omega-E$ for some corresponding n and all ζ on $\Omega-E$ which satisfy $|\zeta-z| <1/n$; by passing to the limit through appropriate points of $\Omega-E$, it therefore holds for *all* ζ on Ω satisfying $|\zeta-z| <1/n$.

Applying Theorem 1 in conjunction with the Baire category theorem, we obtain a neighborhood ω centered at some point a of F and an index N such that $F_N \supset F \cap \omega$. This forces f (analytic on $\omega - F$) to be continuous on ω. Shrinking ω if necessary, we shall suppose that its radius is less than $1/N$. Once again we refer to the work of Meier [9, p. 186] for the fact (easily established by means of the Cauchy integral formula) that the Lipschitz condition

$$\left| f(\zeta) - f(z) \right| \leqq 2N \left| \zeta - z \right|$$

then holds for all ζ, z on the neighborhood ω' about a having radius half that of ω.

With the Lipschitz condition at our disposal it becomes profitable to introduce the areal mean functions $\alpha_r u$ and $\alpha_r v$. That is, we are assured that the integrand in the identity

$$\frac{\alpha_r u(z + h) - \alpha_r u(z)}{h} - \frac{\alpha_r v(z + ih) - \alpha_r v(z)}{h}$$

$$= \frac{1}{\pi r^2} \int_0^r \int_0^{2\pi} \left[\frac{u(z + h + \rho e^{i\theta}) - u(z + \rho e^{i\theta})}{h} - \frac{v(z + ih + \rho e^{i\theta}) - v(z + \rho e^{i\theta})}{h} \right] \rho d\theta d\rho$$

is bounded for $h(\neq 0)$ real and numerically small. Since, as $h \to 0$, the integrand tends to 0 almost everywhere (through values for which the argument belongs to $\Omega-E$, and therefore, by continuity, through all values), there results

$$\partial \alpha_r u / \partial x = \partial \alpha_r v / \partial y.$$

Similarly,

$$\partial \alpha_r u / \partial y = -\partial \alpha_r v / \partial x.$$

The functions $f_r = \alpha_r u + i\alpha_r v$, which have just been shown to be analytic, converge uniformly on compact subsets of ω' to f, so that f must be analytic on ω'. This contradiction proves that f, as extended, is analytic on Ω.

It should be remarked that the proof given here can be somewhat shortened by invoking the classical Looman-Menchoff theorem for f on ω. The development actually adopted is, however, substantially more elementary.

5. Added in proof. The author is indebted to Professor J. Ridder for calling attention to the theorem enunciated in footnote 31 of his paper *Harmonische, subharmonische und analytische Funktionen* [Annali Scuola Normale Superiore, Pisa (2) vol. 9 (1940) pp. 277–287].

This theorem implies that the exceptional set E in Theorem 4 can actually be taken as the union of countably many sets of zero length, not necessarily closed in Ω. However, the derivation outlined by Ridder is based on fairly deep results of Saks and Zygmund, Haslam Jones, and Morera-Rademacher, whereas Theorem 4 follows in an elementary way from Besicovitch's Theorem 1. On the other hand, it is inherent in the nature of the Looman-Menchoff argument that the exceptional set must be an F_σ for this argument to be applicable.

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