SYMMETRIC POLYNOMIALS WITH NON-NEGATIVE COEFFICIENTS

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1. **Introduction.** Brunn [1] proved a theorem on a determinant (an alternant) the elements of which are elementary symmetric functions of positive variables. This theorem reflected by us in a sharper form and applied to polynomials (Theorem 1) is the basis of further investigation. A special case is Theorem 2, important applications of which are Theorems 3 and 4. In §2 we prove these results and in §3 we give some examples. In §4 the foregoing is applied to absolutely monotonic functions; the result is Theorem 5, a generalization of a theorem of Rosenbloom [2]. In §5 an extension of Theorem 3 is deduced (Theorem 6) by considering a function of two variables.

2. Let S_j be the elementary symmetric function of n variables $x_1, x_2, \cdots, x_n$, defined by

$$S_j = \sum x_1 x_2 \cdots x_j \qquad \text{for } j = 1, 2, \cdots, n;$$
$$S_0 = 1;\ S_j = 0 \qquad \text{for } j = -1, -2, \cdots \text{ and } j > n.$$

THEOREM 1. *The determinant* $(S_{k_{ij}})$, $i=1, 2, \cdots, q$; $j=1, 2, \cdots, q$, *with*

$$\begin{aligned} k_{i,m} - k_{i,m+1} &= k(m) > 0, \\ k_{m+1,i} - k_{m,i} &= k^*(m) > 0, \\ &\qquad i = 1, 2, \cdots, q;\ m = 1, 2, \cdots, q-1, \end{aligned} \tag{1}$$

can be written as a symmetric polynomial in $x_1, x_2, \cdots, x_n$ *with non-negative coefficients.*

PROOF. Obviously the determinant in question is a symmetric polynomial in the considered variables.

For $n=1$ the determinant equals zero or unity or a power of x_1. Now applying induction we assume the assertion to be true for $n-1$ and prove the truth for the case n. Therefore we put

$$S_k = S'_k + x_n S'_{k-1}, \tag{2}$$

where S'_k differs from S_k in referring to the variables with x_n left out. Substituting (2) in the determinant we can expand this in increasing powers of x_n, hence

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$$(S_{k_{ij}}) = \sum_{k=0}^{q} A_k x_n^k, \tag{3}$$

where A_k is a sum of determinants the elements of which are elementary symmetric functions of $x_1, x_2, \cdots, x_{n-1}$. The element indices satisfy (1), hence from our assumption it follows that each A_k is a polynomial in $x_1, x_2, \cdots, x_{n-1}$ with non-negative coefficients. Because of (3) our assertion is proved.

An immediate consequence of Theorem 1 is

THEOREM 2. *Let* $\sigma_j = (-1)^j S_j$. *Then the determinant*

$$(\sigma_{m+i-k_j}) \quad (i = 0, 1, \cdots, q; j = 0, 1, \cdots, q; 0 = k_0 < k_1 < \cdots < k_q)$$

multiplied by the factor $(-1)^M$, *where*

$$M = m + (m - k_1) + (m - k_2) + \cdots + (m - k_q) + 1 + 2 + \cdots + q,$$

is expressible as a symmetric polynomial in $x_1, x_2, \cdots, x_n$ *with non-negative coefficients.*

Now we prove the following

THEOREM 3. *Let*

$$f_{h+1}(x) \equiv a_{h0} + a_{h1}x + \cdots + a_{h,n+p}x^{n+p} \quad (h = 0, 1, \cdots, n - 1),$$

where $n \geqq 2$ *and* $p \geqq 0$, *be* n *polynomials with real coefficients such that all determinants* D *of the* n*th order, taken from the matrix* $|a_{ij}|$, $i=0, 1, \cdots, n-1; j=0, 1, \cdots, n+p$, *are non-negative.*

If for n *variables* $x_1, x_2, \cdots, x_n$, *with* $x_i \neq x_j$ *for* $i \neq j$, *we put* $V = V(x) =$ *the determinant* (x_i^j), $i=1, 2, \cdots, n$; $j=0, 1, \cdots, n-1$, *then the expression*

$$(f_i(x_j))/V. \tag{4}$$

can be written as a symmetric polynomial in $x_1, x_2, \cdots, x_n$ *with non-negative coefficients.*

PROOF. From a theorem of Garbieri [3] it follows that (4) is equal to the determinant of $(n+p+1)$th order

$$\begin{aligned} &(B_{ij}),\ B_{ij} = a_{ij} \quad (i = 0, 1, \cdots, n - 1; j = 0, 1, \cdots, n + p), \\ &B_{ij} = \sigma_{i-j} \quad (i = n, n + 1, \cdots, n + p; j = 0, 1, \cdots, n + p), \end{aligned} \tag{5}$$

where σ_j is defined in Theorem 2. By expanding (5) in terms of the $(p+1)$-line minors of the last $p+1$ rows (let i denote the rows) we see that (B_{ij}) is the sum of a number of expressions each of which is a product of a determinant D, the corresponding determinant (σ_{n+i-k_j}), $i=1, 2, \cdots, p+1$; $j=1, 2, \cdots, p+1$; $1\leqq k_1<k_2< \cdots <k_{p+1}\leqq n+p+1$, and the factor $(-1)^S$ where $S=(n+1)+(n+2)+\cdots+(n+p+1)+k_1+k_2+\cdots+k_{p+1}$. Now the exponent M (see Theorem 2) related to this last determinant is equal to $(n+1-k_1)+\cdots+(n+1-k_{p+1})+1+2+\cdots+p$, so that $M\equiv S \pmod 2$. From this conclusion and Theorem 2 the assertion follows.

A special case of Theorem 3 (put $f_h(x)=x^{k_h}$, $h=1, 2, \cdots, n$) is

THEOREM 4 (P. C. ROSENBLOOM [2, p. 459]). *If $k_1, k_2, \cdots, k_n$ are integers with $0\leqq k_1<k_2< \cdots <k_n$, then*

$$(x_i^{k_j})/V; \qquad i, j = 1, 2, \cdots, n, \tag{6}$$

is a symmetric polynomial in $x_1, x_2, \cdots, x_n$ with non-negative coefficients.

REMARK. The last result can also be obtained by application of a theorem of H. Naegelsbach [4].

Acting in this way we find for the expression (6) the determinant

$$\begin{vmatrix} S_n & S_{n-1}\cdot S_{n-k_1+1} & S_{n-k_1-1}\cdot S_{n-k_2+1} & S_{n-k_2-1}\cdots S_{n-k_n+1} \\ 0 & S_n \quad \cdot S_{n-k_1+2} & S_{n-k_1} \quad \cdot S_{n-k_2+2} & S_{n-k_2} \quad \cdots S_{n-k_n+2} \\ \cdot & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0 & 0\cdot\cdots & \cdots\cdots & \cdots\cdot S_1 \end{vmatrix}$$

whose elements satisfy the conditions of Theorem 1.

3. **Examples.** 1. *If for $h=1, 2, \cdots, n$; $p\geqq 0$,*

$$f_h(x) = \sum_{i=0}^{n+p} a_h^{h+i-1} x^i \qquad \textit{with } 0 < a_1 < \cdots < a_n,$$

then $(f_i(x_j))/V$; $i, j=1, 2, \cdots, n$, is a symmetric polynomial in $x_1, x_2, \cdots, x_n$ with non-negative coefficients.

This follows from the fact that the determinants of the nth order taken from the matrix

$$\left| a_i^{i+j-2} \right|, \qquad i = 1, 2, \cdots, n; j = 1, 2, \cdots, n + p + 1,$$

divided by the positive number $V(a)$ are polynomials in the positive $a_1, a_2, \cdots, a_n$ with non-negative coefficients, as follows from Theorem 4.

2. *If the determinants of the nth order taken from the matrix $|a_{ij}|$, $i=1, 2, \cdots, n$; $j=1, 2, \cdots, n+p$, are positive, and if*

$$A_{ij} = \sum_{q=1}^{n+p} \sum_{h=1}^{n} a_{iq} x_h^q x_j^{k_h}, \qquad i, j = 1, 2, \cdots, n,$$

where the k_h are integers with $0 \leqq k_1 < \cdots < k_n$, then $(A_{ij})/V^2$ $(i, j = 1, 2, \cdots, n)$ is a symmetric polynomial with non-negative coefficients.

PROOF. Putting $f_i(x) \equiv \sum_{q=1}^{n+p} a_{iq} x^q$ $(i = 1, 2, \cdots, n)$, we have

$$A_{ij} = \sum_{h=1}^{n} x_j^{k_h} \sum_{q=1}^{n+p} a_{iq} x_h^q = \sum_{h=1}^{n} x_j^{k_h} f_i(x_h),$$

so that

$$(A_{ij}) = (x_i^{k_j})(f_i(x_j)) \qquad (i, j = 1, 2, \cdots, n).$$

Application of Theorem 3 completes the proof.

4. THEOREM 5. *If $f(x)$ and $g(x)$ are power series with non-negative coefficients converging in the interval $0 \leqq x < a$, then the expression*

$$(7) \quad (-1)^{n-1} \det \left| 1 x_i^1 \cdots x_i^{n-3} f(ux_i) g(vx_1 \cdots x_{i-1} x_{i+1} \cdots x_n) \right| / V,$$
$$i = 1, 2, \cdots, n,$$

can be written as a power series in the variables $x_1, x_2, \cdots, x_n, u, v$ with non-negative coefficients converging in the range

$$0 \leqq x_1, x_2, \cdots, x_n < a; \; 0 \leqq u \leqq 1; \; 0 \leqq v \leqq \frac{1}{a^{n-1}} \cdot$$

PROOF. Putting

$$f(x) = \sum_{q=0}^{\infty} a_q x^q, \qquad g(x) = \sum_{m=0}^{\infty} b_m x^m, \qquad a_q \geqq 0, \; b_m \geqq 0,$$

then (7) can be expressed as

$$(-1)^{n-1} \sum_{q=0}^{\infty} a_q u^q \left| 1 x_i^1 \cdots x_i^{n-3} x_i^q g(vx_1 \cdots x_{i-1} x_{i+1} \cdots x_n) \right| / V$$

$$= (-1)^{n-1} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} a_q b_m u^q v^m \left| 1 x_i^1 \cdots x_i^{n-3} x_i^q (x_1 \cdots x_{i-1} x_{i+1} \cdots x_n)^m \right| / V$$

$$= (-1)^{n-1} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} a_q b_m u^q v^m \left| x_i^m x_i^{m+1} \cdots x_i^{m+n-3} x_i^{m+q} 1 \right| / V$$

$$= \sum_{q=n-2}^{\infty} \sum_{m=0}^{\infty} a_q b_m u^q v^m \left| 1 x_i^m x_i^{m+1} \cdots x_i^{m+n-3} x_i^{m+q} \right| / V,$$

so that from Theorem 4 the correctness of this theorem follows.

REMARK. For $n=3$ this theorem is a result of Rosenbloom about absolutely monotonic functions [2]. (The range of u and v given there is not quite correct.)

5. THEOREM 6. *Let $F(x, y)$ be the function $\sum_{i=0}^{n+p} \sum_{j=0}^{n+p} a_{ij} x^i y^j$ $(p \geqq 0)$, with the property that all determinants D of the nth order taken from the matrix $|a_{ij}|$ are non-negative. Let $(x_1, x_2, \cdots, x_n)$ and $(y_1, y_2, \cdots, y_n)$ with $x_i \neq x_j$, $y_i \neq y_j$ $(i \neq j)$ be two sets of variables.*

Then the expression

$$(F(x_i, y_j))/V(x)V(y). \tag{8}$$

is a polynomial, symmetric in $x_1, x_2, \cdots, x_n$ as well as in $y_1, y_2, \cdots, y_n$ with non-negative coefficients.

PROOF. On account of another theorem of Garbieri [3] the expression (8) is identical with

$$(-1)^{p-1}(t_{ij}), \qquad i, j = 0, 1, \cdots, n + 2p + 1, \tag{9}$$

where

$$t_{ij} = a_{ij} \qquad (i, j = 0, 1, \cdots, n + p)$$

$$= \sigma'_{j-i-p-1} \qquad (i=0, 1, \cdots, n+p, j=n+p+1, \cdots, n+2p+1)$$

$$= \sigma_{i-j-p-1} \qquad (i=n+p+1, \cdots, n+2p+1, j=0, 1, \cdots, n+2p+1),$$

where $(-1)^j\sigma_j$ and $(-1)^j\sigma'_j$ are the elementary symmetric functions of $x_1, x_2, \cdots, x_n$ and $y_1, y_2, \cdots, y_n$ respectively. We develop the determinant in (9) in terms of the $(p+1)$-line minors of the last $p+1$ rows. The term corresponding with the minor indicated by the column-indices $k_1, k_2, \cdots, k_{p+1}$, say $\mathfrak{M}$, possesses the sign

$$(-1)^{(n+p+2)+(n+p+3)+\cdots+(n+2p+2)+k_1+k_2+\cdots+k_{p+1}} = (-1)^M.$$

If in $\mathfrak{M}$ we replace each σ_j by $(-1)^j s_j$, then the new minor $\mathfrak{M}'$ has the sign

$$(-1)^{(n-k_1+1)+(n-k_2+1)+\cdots+(n-k_{p+1}+1)+1+2+\cdots+p} = (-1)^N.$$

The complementary minor of $\mathfrak{M}$ with elements a_{ij} and σ'_k, say $\overline{\mathfrak{M}}$, can be expanded in terms of the $(p+1)$-line minors of the last $p+1$ columns. The term in the expansion of $\overline{\mathfrak{M}}$ corresponding with the minor $\mathfrak{N}$ indicated by the row-indices $q_1, q_2, \cdots, q_{p+1}$ is provided with the sign

$$(-1)^{(n+1)+(n+2)+\cdots+(n+p+1)+q_1+q_2+\cdots+q_{p+1}} = (-1)^P.$$

In $\mathfrak{N}$ again we replace the σ' by S', and the new minor $\mathfrak{N}'$ has the sign

$$(-1)^{(n-q_1+1)+(n-q_2+1)+\cdots+(n-q_p+1+1)+1+2+\cdots+p} = (-1)^Q.$$

Thus the determinant in (9) is the sum of terms each of which is a product of 3 determinants $\mathfrak{M}'$, $\mathfrak{N}'$, D and the factor $(-1)^{M+N+P+Q}$. By simple calculation we see that

$$M + N + P + Q \equiv (p + 1)^2 \pmod 2.$$

As from our assumption the determinants D are non-negative, it follows that each term in the development of (9) has the positive sign, on account of $(-1)^{(p+1)+(p+1)^2} = 1$.

The application of Theorem 1 to each $\mathfrak{M}'$ and $\mathfrak{N}'$ completes the proof.

References

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