A NOTE ON A THEOREM OF ROTH

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In a recent paper Roth [1] proved:

THEOREM 1. *If A and B are $n \times n$ matrices with elements in the field F, whose characteristic polynomials are*

$$a_0(x^2) - xa_1(x^2) \quad \textit{and} \quad b_0(x^2) - xb_1(x^2)$$

respectively, where $a_0(x)$, $a_1(x)$, $b_0(x)$, $b_1(x)$ are elements in the polynomial domain, $F[x]$, of F; and if the rank of $A-B$ does not exceed unity; then the characteristic polynomial of AB is

$$(-1)^n[a_0(x)b_0(x) - xa_1(x)b_1(x)].$$

In this note the following extension of this theorem will be proved:

THEOREM 2. *If A and B are as given in Theorem 1 and k is any element of the field F, then the characteristic polynomial of $AB+k(A-B)$ is*

$$(-1)^n\{a_0(x)b_0(x) - xa_1(x)b_1(x) + k[a_0(x)b_1(x) - a_1(x)b_0(x)]\}.$$

Use the notation of Roth and write $A-B=D=R^TS$ where R and S are row vectors in n-space.

Since

$$|Ix - A| = a_0(x^2) - xa_1(x^2) \quad \text{and} \quad |Ix - B| = b_0(x^2) - xb_1(x^2)$$

we have $|Ix+A| = (-1)^n[a_0(x^2)+xa_1(x^2)]$.

It then follows that

$$\begin{aligned} |Ix^2 - AB + x(A - B)| = (-1)^n\{&a_0(x^2)b_0(x^2) - x^2a_1(x^2)b_1(x^2) \\ &+ x[a_1(x^2)b_0(x^2) - a_0(x^2)b_1(x^2)]\}. \end{aligned} \tag{1}$$

From Lemma II of Roth's paper, it follows that

$$|Ix^2 - AB + x(A - B)| = |Ix^2 - AB| + xS[\operatorname{adj}(Ix^2 - AB)]R^T. \tag{2}$$

The right members of (1) and (2) are identically equal and each is written as the sum of an even polynomial and an odd polynomial. Equating corresponding parts gives

$$|Ix^2 - AB| = (-1)^n[a_0(x^2)b_0(x^2) - x^2a_1(x^2)b_1(x^2)] \tag{3}$$

and

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$$(4)\quad S[\operatorname{adj}(Ix^2 - AB)]R^T = (-1)^n[a_1(x^2)b_0(x^2) - a_0(x^2)b_1(x^2)].$$

Also from Lemma II (Roth), it follows that

$$(5)\quad \begin{aligned} |Ix^2 - [AB + k(A - B)]| = {} & |Ix^2 - AB| \\ & - kS[\operatorname{adj}(Ix^2 - AB)]R^T. \end{aligned}$$

Replacing x^2 by y and substituting from (3) and (4) into (5) gives

$$(6)\quad \begin{aligned} |Iy - [AB + k(A - B)]| = {} & (-1)^n\{a_0(y)b_0(y) - ya_1(y)b_1(y) \\ & + k[a_0(y)b_1(y) - a_1(y)b_0(y)]\}. \end{aligned}$$

This completes the proof of the theorem.

The characteristic polynomials of many other functions of A and B may now be written at once. Since $(A-B)^u = (SR^T)^{u-1}(A-B)$ for every positive integer u, the characteristic polynomial of $AB + c(A-B)^u$ may be obtained from (6) by taking $k = c(SR^T)^{u-1}$. In particular for $c=1$ and $u=2$ this becomes $AB+(A-B)^2 = A^2 - BA + B^2$. Thus (6) may be used to write the characteristic polynomial of $A^2 - BA + B^2$.

Also Lemma II gives the characteristic polynomial of $A - B$ to be $x^n - SR^Tx^{n-1}$. Hence, the matrix R^TS is nilpotent if and only if the vectors R and S are orthogonal.

Reference

W. E. Roth, *On the characteristic polynomial of the product of two matrices*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 1–3.

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