ment a which suffice. Let $F=B(t)$, B any field of characteristic two and t transcendental over B. If $f(t)$ denotes an arbitrary element of $B(t)$, then define α by $f(t)\alpha=f(1/t)$, and let $a=t+1/t$.

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ON THE CHARACTERISTIC FUNCTION OF A MATRIX PRODUCT

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In a recent note [1], Roth has proved this result.

Theorem 1. *Let A and B be $n\times n$ matrices, with elements in a field F, and let*

$$|xI - A| = a_0(x^2) - xa_1(x^2), \qquad |xI - B| = b_0(x^2) - xb_1(x^2),$$

where a_0, a_1, b_0, and b_1 are elements in the polynomial ring $F[x]$. If the rank of $A-B$ is not greater than unity, then

$$|xI - AB| = (-)^n[a_0(x)b_0(x) - xa_1(x)b_1(x)].$$

In his proof, which is essentially a verification, Roth derives some interesting but unnecessary information. Here I present a proof which is shorter, direct, and leads naturally to a more general result involving three matrices.

The essential step in my proof is the observation that if A is a nonsingular matrix and M is a matrix of rank 1, then

$$|A+M| = |A| + \sum \Delta_i$$

where $\sum\Delta_i$ is a sum of n determinants, each consisting of $n-1$ columns of A and *one* column of M. This follows from the fact that, M being of rank 1, any two columns of M are linearly dependent.

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For the case at hand we have

$$\begin{aligned} |xI - A|\,|xI + B| &= |(xI - A)(xI + B)| \\ &= |x^2I - AB - x(A - B)| \end{aligned}$$

and this determinant is equal to $|x^2I-AB|$ if $A-B$ has zero rank, while if $A-B$ has rank 1, we have

$$|x^2I - AB - x(A - B)| = |x^2I - AB| - x\sum \Delta_i,$$

where each determinant Δ_i has $n-1$ columns chosen from x^2I-AB and one column from $A-B$. It is observed that the terms of $x\sum\Delta_i$ contain only *odd* powers of x. Thus, in either case, $|x^2I-AB|$ is equal to the *even* part of $|xI-A|\,|xI+B|$. Now

$$|xI - A|\,|xI + B| = (-)^n[a_0(x^2) - xa_1(x^2)][b_0(x^2) + xb_1(x^2)],$$

and the even part is $(-)^n[a_0(x^2)b_0(x^2)-x^2a_1(x^2)b_1(x^2)]$. Hence, writing $y=x^2$, we have

$$|yI - AB| = (-)^n[a_0(y)b_0(y) - ya_1(y)b_1(y)],$$

and this is Roth's result.

Before extending this result we prove the

LEMMA. *If H and K are nonzero square matrices, such that $xH-K$ is of rank* 1, *for x indeterminate over the field F, then either*

(i) $H=\boldsymbol{u}\boldsymbol{h}'$, $K=\boldsymbol{u}\boldsymbol{k}'$,

or

(ii) $H=\boldsymbol{u}\boldsymbol{h}'$, $K=\boldsymbol{v}\boldsymbol{h}'$,

where $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{h}$, $\boldsymbol{k}$ are column vectors. Conversely, if H and K satisfy (i) *and* (ii) *then $xH-K$ is of rank* 1.

PROOF. Since $xH-K$ is of rank 1 for all x, it follows that H and K are each of rank 1 and hence are of the form

$$H = \boldsymbol{u}\boldsymbol{h}', \qquad K = \boldsymbol{v}\boldsymbol{k}',$$

where $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{h}$, $\boldsymbol{k}$ are column vectors. If we now equate to zero all the two-rowed minors of $xH-K$, it is easily found that either $\boldsymbol{u}=\boldsymbol{v}$ or $\boldsymbol{h}=\boldsymbol{k}$, and this proves the lemma. The converse is obviously true.

From this lemma we proceed to

THEOREM 2. *Let A_1, A_2, and A_3 be $n\times n$ matrices, such that*

$$|xI - A_i| = a_{0i}(x^3) + xa_{1i}(x^3) + x^2a_{2i}(x^3) \qquad (i = 1, 2, 3)$$

and write $H=A_1+A_2+A_3$, $K=A_1A_2+A_1A_3+A_2A_3$. If H and K satisfy the lemma, or if $H=K=0$, then

$$|xI - A_1A_2A_3| = a_{01}a_{02}a_{03} + x[a_{11}(a_{02}a_{23} + a_{03}a_{22}) + a_{12}(a_{01}a_{23} + a_{03}a_{21}) + a_{13}(a_{01}a_{22} + a_{02}a_{21})] + x^2a_{21}a_{22}a_{23},$$

where $a_{ij} = a_{ij}(x)$.

PROOF. We have

$$(xI - A_1)(xI - A_2)(xI - A_3) = x^3I - A_1A_2A_3 - x(xH - K).$$

If $H = K = 0$ we have

$$E \equiv |xI - A_1|\,|xI - A_2|\,|xI - A_3| = |x^3I - A_1A_2A_3|.$$

If $xH - K$ is of rank 1 for all x, we have

$$E = |x^3I - A_1A_2A_3| - x\sum\Delta_i,$$

where each determinant Δ_i, since it consists of $n-1$ columns of $x^3I - A_1A_2A_3$ and 1 column of $xH - K$, expands into a polynomial each term of which involves x to the power $3k$ or $3k+1$ for some integer k. Now $x\sum\Delta_i$ is a polynomial, each term of which involves x to a power $3k+1$ or $3k+2$. Thus, in either case, $|x^3I - A_1A_2A_3|$ is equal to the sum of the terms of $|xI - A_1|\,|xI - A_2|\,|xI - A_3|$ which involve powers of x^3. If we pick out these terms and replace x^3 by x the result follows.

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