ON UNIVERSALLY INTEGRABLE FUNCTIONS¹

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Let X be an abstract space and $\mathfrak{A}$ be a Boolean algebra of sets E in X . Let ρ be a finite collection of disjoint sets $\{E_k\}$ in $\mathfrak{A}$ which partition X . The partitions ρ form a lattice [1] if we define $\rho < \rho'$ whenever ρ' is a refinement of ρ . Hence, we can use the partitions $\{\rho\}$ to index nets of real numbers and can take limits with respect to ρ in the Moore-Smith sense [3].

If F is a real-valued, finitely additive measure on $\mathfrak{A}$ and f is a real-valued function on X , we can define an integral

$$(1) \quad \int f dF = \lim_{\rho} \sum_{\rho} f(x_k) F(E_k)$$

whenever this limit with respect to $\rho = \{E_k\}$ exists independently of the choice of x_k in E_k , and is finite. We call f *universally integrable* if the integral (1) exists for every finitely additive measure F on $\mathfrak{A}$. The purpose of this note is to characterize universally integrable functions.

A *step function* is any finite linear combination of characteristic functions of sets in $\mathfrak{A}$. For each step function g there exists a partition $\rho = \{E_k\}$ such that

$$(2) \quad g(x) = \sum_{\rho} a_k E_k(x)$$

where

$$(3) \quad E(x) = \begin{cases} 0 & \text{if } x \text{ is not in } E, \\ 1 & \text{if } x \text{ is in } E, \end{cases}$$

the characteristic function of E . We denote the oscillation of f on E by $|f(E)|$. That is,

$$(4) \quad |f(E)| = \sup_{x \in E} f(x) - \inf_{x \in E} f(x).$$

LEMMA. f is a uniform limit of step functions if, and only if,

$$(5) \quad \lim_{\rho} \max_k |f(E_k)| = 0$$

where $\rho = \{E_k\}$.

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PROOF. If f is a uniform limit of step functions, then for arbitrary $\alpha > 0$ there exists g of the form (2) such that $|f(x) - g(x)| < \alpha/2$ for all x . In particular, for x in E_k , $|f(x) - a_k| < \alpha/2$. Thus, for x and y in E_k , $|f(x) - f(y)| \leq |f(x) - a_k| + |a_k - f(y)| < \alpha$. Therefore, $|f(E_k)| < \alpha$ for all E_k in ρ . Moreover, $|f(E)| < \alpha$ for E belonging to any refinement of ρ . Hence (5).

Conversely, (5) implies that for arbitrary $\alpha > 0$ there exists ρ sufficiently fine that $|f(E_k)| < \alpha$ for all E_k in ρ . Choosing any x_k in E_k , let $a_k = f(x_k)$ in (2). Then $|f(x) - g(x)| < \alpha$ for all x . So f is a uniform limit of step functions.

THEOREM. *A function is universally integrable if, and only if, it is a uniform limit of step functions.*

PROOF. Suppose f were not a uniform limit of step functions. Then, there exists $\alpha > 0$ such that every partition $\rho = \{E_k\}$ has some E_k for which $|f(E_k)| > \alpha$. For each ρ , let E_ρ be the union of all such E_k in ρ . Then, $E_\rho \neq \emptyset$ for all ρ . Moreover, if $\rho < \rho'$, then $E_\rho \supseteq E_{\rho'}$. Hence $\{E_\rho\}$ has the finite intersection property: $E_{\rho_1} \cdot \cdot \cdot E_{\rho_n} \neq \emptyset$. This follows from the fact that $E_{\rho_1} \cdot \cdot \cdot E_{\rho_n} \supseteq E_\rho$ where ρ is any common refinement of $\rho_1, \cdot \cdot \cdot, \rho_n$. With Zorn's lemma we imbed $\{E_\rho\}$ in a maximal class M of sets in $\mathfrak{A}$ having the finite intersection property. Define

$$(6) \quad F(E) = \begin{cases} 0 & \text{if } E \text{ is not in } M, \\ 1 & \text{if } E \text{ is in } M. \end{cases}$$

If $AB = \emptyset$, then not both A and B can belong to M . Also, if A belongs to M and $E \supset A$, then E belongs to M . Hence, $F(A+B) = F(A) + F(B)$. So F is a finitely additive measure. Finally, $F(E_\rho) = 1$ for all ρ . Thus, for all ρ ,

$$(7) \quad \sum_{\rho} |f(E_k)| F(E_k) > \alpha.$$

Hence, the integral (1) does not exist for F defined by (6).

So f must be a uniform limit of step functions. Since step functions are bounded, this implies that f is bounded.

Conversely, if f is a uniform limit of step functions, then for each $\alpha > 0$ there exists a partition $\rho_\alpha = \{A_m\}$ such that $|f(A_m)| < \alpha$ for all A_m in ρ_α . Hence, for all $\rho \geq \rho_\alpha$ and all finitely additive measures F ,

$$(8) \quad \sum_{\rho} |f(E_k)| F(E_k) < \alpha F(X).$$

Thus,

$$(9) \quad \lim_{\rho} \sum_{\rho} |f(E_k)| F(E_k) = 0,$$

and (1) exists. (See [2].)

For a σ -algebra we call f universally σ -integrable if the integral (1) exists for all countably additive measures F . It can be readily verified that such a function f must be bounded and, for each measure F , must be F -equivalent to some measurable function. The necessity for the measurability of f itself is an open question. An example of a universally σ -integrable function that is not measurable would be quite interesting.

REFERENCES

1. G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, New York, 1940.
2. T. H. Hildebrandt, *On bounded linear functional operations*, Trans. Amer. Math. Soc. vol. 36 (1934) pp. 868–875.
3. E. H. Moore and H. L. Smith, *A general theory of limits*, Amer. J. Math. vol. 44 (1922) pp. 102–121.

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